MA 294: Applied Abstract Algebra / Spring 2022
Exercise sheet #2: equivalence relations
Solution notes on second page

(1) For each relation R on the set X below, determine whether it is reflexive, symmetric, and transitive. If the relation is an equivalence relation, describe the equivalence classes.

(a) $X = \mathbb{Z}$ and xRy if $x = y$

(b) $X = \mathbb{R}$ and xRy if $x \leq y$

(c) $X = \mathbb{R}$ and xRy if $|x - y| = 5$

(d) $X = \mathbb{Z}$ and xRy if $3 \mid (x - y)$

(e) $X = \mathbb{Z}$ and xRy for no $x, y \in \mathbb{Z}$

(f) $X = \mathbb{R}$ and xRy if $xy > 0$

(g) $X = \mathbb{R}$ and xRy if $xy \geq 0$

(h) $X = \mathbb{R}$ and xRy if $xy > 0$ or $x = y = 0$

(i) $X = \mathbb{R}$ and xRy if $x - y \in \mathbb{Z}$

(j) $X = \mathbb{R}^2$ and $(a, b)R(c, d)$ if for some $\lambda \in \mathbb{R}$ we have $(c, d) = (\lambda a, \lambda b)$

(k) $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$ and $(a, b)R(c, d)$ if $ad = bc$

(2) Let $\sim$ be an equivalence relation on a set X. Recall that for $a \in X$, the *equivalence class* $[a]$ of a with respect to $\sim$ is the set

$$[a] = \{b : b \sim a\}.$$

Show that any two equivalence classes are either disjoint or coincide. Show that the equivalence classes *partition* X: that is, we have a decomposition of X into a disjoint union of equivalence classes.

(3) **Going further:** Suppose the set X is a disjoint union of nonempty subsets S_i for i in some indexing set I: that is, the S_i are all nonempty subsets of X with the property that $S_i \cap S_j = \varnothing$ if $i \neq j$, and moreover $X = \bigcup_{i \in I} S_i$.

Show that there is an equivalence relation $\sim$ on X so that the S_i are the equivalence classes with respect to $\sim$.

(1) (a) ER. Each equivalence class contains exactly one element of $\mathbb{Z}$.
  (b) reflexive and transitive, but not symmetric
  (c) symmetric but not reflexive or transitive
  (d) ER. The equivalence classes are the three residue classes modulo 3.
  (e) symmetric and transitive but not reflexive
  (f) transitive and symmetric but not reflexive
  (g) reflexive and symmetric but not transitive
  (h) ER. Three equivalence classes: positive reals, negative reals, and 0 in its own class.
  (i) ER. There's one equivalence class for every element α of $[0, 1)$; the equivalence classe of α is $\alpha + \mathbb{Z}$.
  (j) reflexive and symmetric but not transitive — $\lambda = 0$ causes problems. (How can you fix this to make an ER?)
  (k) ER. The equivalence classes correspond to equivalent fractions; there's one for every element of $\mathbb{Q}$.

(2) See Theorem 7.3