MA 294: Applied Abstract Algebra / Spring 2022
Exercise sheet #3: cyclic groups, their generators, and their subgroups

Recall that a *generator* of a cyclic group G is an element $a \in G$ so that $G = \langle a \rangle$.

(1) In class we found all the generators of $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$.

(a) Find all the generators of $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, and $\mathbb{Z}_{10}$.

(b) Nika guessed that maybe $[a]$ generates $\mathbb{Z}_m$ if and only if a does not divide m. Is this statement true? If it is true, explain why. If it is false, salvage this statement (that is, fix it) — and then prove the corrected statement!

(2) (a) Which of the groups $\mathbb{Z}_3^\times$, $\mathbb{Z}_4^\times$, $\mathbb{Z}_5^\times$, $\mathbb{Z}_6^\times$, $\mathbb{Z}_7^\times$, $\mathbb{Z}_8^\times$, $\mathbb{Z}_9^\times$, $\mathbb{Z}_{10}^\times$ are cyclic?

(b) What about $\mathbb{Z}_m^\times$ for $m \leq 20$? $m \leq 30$?

(c) Any thoughts or guesses about the m for which $\mathbb{Z}_m^\times$ is cyclic?

(3) (a) Is $\mathbb{Z}_3 \times \mathbb{Z}_4$ cyclic? If so, construct an isomorphism $\mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4$. If not, explain why such an isomorphism is impossible.

(b) Is $\mathbb{Z}_2 \times \mathbb{Z}_6$ cyclic? If so, construct an isomorphism $\mathbb{Z}_{12} \to \mathbb{Z}_2 \times \mathbb{Z}_6$. If not, explain why such an isomorphism is impossible.

(4) (a) Find all the subgroups of $\mathbb{Z}_{10}$.

(b) Find all the subgroups of $\mathbb{Z}_{12}$.

(c) Find all the subgroups of $\mathbb{Z}_9^\times$.

(d) Find all the subgroups of $\mathbb{Z}_{13}^\times$.