MA 294: Applied Abstract Algebra / Spring 2022
Homework assignment #4
Due Thursday 2/17/2022 by 4pm

New stuff in blue!

Turn in your work either in class or before 4pm in the envelope hanging on MCS 127.

- Please staple or otherwise connect the pages of your work. There is a stapler in the math department main office.
- Write your name on the front page. Plan ahead: perhaps you need to carry around a pen for this purpose on Thursdays?
- Consider using a pen rather than a pencil, especially if your pencilwork is smudgy.
- **Challenge problems:** Challenge problems are optional. Please write up your solutions to challenge problems separately. Alternatively, come tell me your solution during office hours. You may also do this later, anytime during the semester.

Recall that $\mathbb{Z}_m^\times = U_m = \{[a] \in \mathbb{Z}_m : \text{there exists } [b] \text{ in } \mathbb{Z}_m \text{ with } [a][b] = [1]\}$. This is an abelian group of order $\varphi(m)$. We showed in class an equivalent description:
$\mathbb{Z}_m^\times = \{[a] \in \mathbb{Z}_m : \gcd(a, m) = 1\}$.

Read section 13.3 through the proof of Theorem 13.3.1 and section 10.3 through p. 95.

(1) Find all $m \leq 20$ so that $\varphi(m) = 4$. For each of these m, find the group structure of $\mathbb{Z}_m^\times$. Is $\mathbb{Z}_m^\times$ isomorphic to $\mathrm{Symm}(\square)$ or to $\mathrm{Rot}(\square)$ in each case? Explain.

(2) Find all $m \leq 20$ so that $\varphi(m) = 6$. For each of these m, find the group structure of $\mathbb{Z}_m^\times$. Is $\mathbb{Z}_m^\times$ isomorphic to $\mathbb{Z}_6$? To $\mathrm{Symm}(\triangle)$? Explain.

Read sections 7.2 and 7.3 on equivalence relations.

(3) Exercise 20.5.3

Read section 20.4

(4) Exercise 20.4.3

(5) If a is an element of a group, show that $\mathrm{ord}(a) = \mathrm{ord}(a^{-1})$. Don't neglect the case where the orders are infinite!

(6) Let G be a finite abelian group of order n. Show that $a^n = 1$ for any $a \in G$ as follows.
  (a) Consider the sets $G = \{g : g \in G\}$ and $aG = \{ag : g \in G\}$.
  What can you say about these two sets? Explain.
  (b) Compare the product of the elements of G to the product of the elements of aG.
  (c) Conclude that $a^n = 1$. Explain.
  (d) What does this imply about the order of a? Explain!

Challenge question

(7) Exercise 12.2.5