**MA 294: Applied Abstract Algebra / Spring 2022**
**Homework assignment #6 final version**
~~**(preliminary version: more problems will be added 3/17/2022)**~~
**Due Thursday 3/24/2022 by 4pm**

Turn in your work either in class or before 4pm in the envelope hanging on MCS 127.

- Please staple or otherwise connect the pages of your work. There is a stapler in the math department main office. Yes, this requires a tiny bit of planning — but I know we can do this! Our grader is anxious that homework set pages might get lost.
- Write your name on the front page. Plan ahead: perhaps you need to carry around a pen for this purpose on Thursdays?
- Consider using a pen rather than a pencil, especially if your pencilwork is smudgy.

Read section 20.8 through the middle of p. 275.

(1) Exercise 20.8.2. In the notation we've been using in class, $H = \{1, \text{flip}(|)\}$.

(2) Exercise 20.8.3. It may be helpful to introduce the notation $r = \text{rot}(72°)$ and $f = \text{flip}(|)$. (Note that in order for $\text{flip}(|)$ to be a symmetry, we have to assume that the pentagon has a horizontal side.)

Read section 10.6.

(3) Exercise 10.6.1

(4) Exercise 10.6.2

More questions

(5) Let $p$ and $q$ be two distinct primes. Prove that $\varphi(pq) = (p-1)(q-1)$.

(Do this by hand, by thinking about the numbers less than $pq$ that are relatively prime to $pq$. Do not use any theorems about the multiplicativity of $\varphi$ that we have not discussed in this course.)

(6) Consider the following $2 \times 2$ matrices with coefficients in $\mathbb{C}$:

$$\mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(a) Show that the set $Q_8 := \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ is closed under matrix multiplication by constructing a multiplication table. Here $1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $-1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

(b) Use this table to prove that $Q_8$ is a group.

(c) We've found four groups of order 8 up to isomorphism so far:

$$\mathbb{Z}_8, \qquad \mathbb{Z}_4 \times \mathbb{Z}_2, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \text{Symm}(\square).$$

Is $Q_8$ isomorphic to any of these? If not explain why not; if yes, construct an explicit isomorphism.

(7) Let $G$ be a group, and suppose that $a, b \in G$ are commuting elements so that the cyclic subgroups they generate intersect trivially: that is, $ab = ba$ and $\langle a \rangle \cap \langle b \rangle = \{1\}$. Prove that $\operatorname{ord}(ab) = \operatorname{lcm}\big(\operatorname{ord}(a)\operatorname{ord}(b)\big)$.

(8) (a) Let $G$ be a group and suppose $a, b \in G$ are commuting elements so that $\operatorname{ord}(a)$ and $\operatorname{ord}(b)$ are relatively prime. Show that $\operatorname{ord}(ab) = \operatorname{ord}(a)\operatorname{ord}(b)$.

(b) Give an example to show that $\operatorname{ord}(ab) \neq \operatorname{ord}(a)\operatorname{ord}(b)$ in general, even if $\operatorname{ord}(a)$ and $\operatorname{ord}(b)$ are relatively prime.

(9) Show that any group of even order has an element of order 2.

(*Hint:* pair up elements with their inverses. Which elements are paired with themselves?)

(10) Let $G$ be a group of order $2n$, where $n > 1$ is odd. Show that $G$ must have an element of order strictly greater than 2.

(*Hint:* Suppose every element $g$ of $G$ satisfies $g^2 = 1$. Show that $G$ must have two elements $a$, $b$ that are distinct from each other and from the identity. Show that $\langle a, b \rangle$ is a subgroup of $G$ isomorphic to the Klein 4-group and derive a contradiction. Problem (3) on HW #3 might be helpful as a way of setting the scene.)

(11) Show that any group of order 6 is either cyclic or isomorphic to $\operatorname{Symm}(\triangle)$.