MA 294: Applied Abstract Algebra / Spring 2022
Homework assignment #8
Due Thursday 4/7/2022 by 4pm

Turn in your work either in class or before 4pm in the envelope hanging on MCS 127.

- Please staple or otherwise connect the pages of your work. There is a stapler in the math department main office. Yes, this requires a tiny bit of planning — but I know we can do this! Our grader is anxious that homework set pages might get lost.
- Write your name on the front page. Plan ahead: perhaps you need to carry around a pen for this purpose on Thursdays?
- Consider using a pen rather than a pencil, especially if your pencilwork is smudgy.

Read sections 21.2 and 21.3. Recall that for a group G acting on a set X containing an element x, the orbit of x is $Gx = \mathrm{Orb}(x)$; and the stabilizer group of x is $G_x = \mathrm{Stab}(x)$.

(1) Exercise 21.2.2

(2) Exercise 21.2.3

(3) Exercise 21.2.5

(4) Let H be a subgroup of a group G, and g and element of G. Show that the subset

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

is also a subgroup of G. This subgroup is said to be *conjugate* to H.

(5) Now let G be a group of permutations of a set X, and suppose that $x, y \in X$ are in the same orbit. Show that $\mathrm{Stab}(x)$ and $\mathrm{Stab}(y)$ are conjugate subgroups.

More precisely, if $g \in G(x \to y)$, then show that $\mathrm{Stab}(y) = g\,\mathrm{Stab}(x)g^{-1}$.

(6) Let G be the group of rotational symmetries of a cube.

  (a) View G as a group of permutation of the set V of vertices of the cube. Let $v \in V$ be a vertex. What's the size of $\mathrm{Orb}(v)$? What's the size of $\mathrm{Stab}(v)$? Explain.

  (b) View G as a group of permutations of the set F of faces of the cube. Let $f \in F$ be a face. What's the size of $\mathrm{Orb}(f)$? What's the size of $\mathrm{Stab}(f)$? Explain.

  (c) View G as a group of permutations of the set E of edges of the cube. Let $e \in E$ be an edge What's the size of $\mathrm{Orb}(e)$? What's the size of $\mathrm{Stab}(e)$? Explain.

  (d) What is the order of G? Explain.

A *diagonal* of a cube connects two opposite vertices. (The line connecting opposite vertices goes through the center of the cube.)

(7) Now view the same group G from (6) as a group of permutations of the set D of diagonals of the cube. Let $d \in D$ be a diagonal. What is the size of $\mathrm{Orb}(d)$? What is the size of $\mathrm{Stab}(d)$? Explain what each element of $\mathrm{Stab}(d)$ does to the cube and identify the group structure of $\mathrm{Stab}(d)$.