MA 541: Modern Algebra I / Fall 2019
Homework assignment # 11
Due WEDNESDAY 12/11/2019 at 4:45pm

12/5/2019: Minor clarification edit on the hint for 2g.
12/9/2019: In 4 we must assume $n \neq 1$. Also, clarification about the word "cube" for 2.

(0) Read in F: sections 16, 36, 37. You may skip the proofs of the Sylow theorems if you wish.

(1) Prove that there are no nonabelian simple groups of order $n < 60$. We did (or will do) this in class for $n \leq 25$. Recall that we have proved the following lemmas:

- Lemma P: If G is a group of prime order p, then G is a simple group isomorphic to $\mathbb{Z}_p$.
- Lemma P^n:
  If G is a group of order p^n, where p is prime and $n \geq 2$, then G is not simple.
- Lemma PQ:
  If G is a group of order pq, with p and q are distinct primes, then G is not simple.

Feel free to write new lemmas of your own. I encourage you to work with other students on this problem!

Optional challenge: Can you extend to groups of order $n < 168$ except $n = 60$?
(The group $\mathrm{PSL}_2(\mathbb{Z}_7) = \mathrm{SL}_2(\mathbb{Z}_7)/\{\pm 1\}$ is simple of order 168.)

(2) Let G be the group of rotational symmetries of a cube. Let C be the points of the cube, so that G acts on C. (Note: *cube* usually refers to the solid three-dimensional object, so C is intended to be all the points of the cube, including the interior. But you may do the question with C being the surface of the cube so long as you're clear about what you mean.)

(a) Let $v \in C$ be a vertex of the cube. What is the orbit of v under the action of G? Describe the stabilizer of v. How many elements in each?

(b) Let $f \in C$ be the center of a face of the cube. What is the orbit of f under the action of G? Describe the stabilizer of f. How many elements in each?

(c) Let $e \in C$ be the midpoint of one of the edges of the cube. What is the orbit of e under the action of G? Describe the stabilizer of e. How many elements in each?

(d) What is the order of G?

(e) Does the action of G on C have any fixed points? Explain.

(f) Are there any points in C whose stabilizer is trivial? Explain.

(g) **Optional challenge part:** Prove that the group of all symmetries of the cube is isomorphic to $G \times \mathbb{Z}_2$.
(*Hint:* Position the cube in $\mathbb{R}^3$ with its center at the origin and its vertices at $(\pm 1, \pm 1, \pm 1)$, and consider the map $\vec{v} \mapsto -\vec{v}$.)

(3) (a) Suppose G is a finite group with primes p and q dividing $|G|$. If G has a normal p-Sylow subgroup P and a normal q-Sylow subgroup Q, show that the elements of P commute with all the elements of Q.

(b) Let G be a finite group. Show that every p-Sylow subgroup of G is normal if and only if G is the direct product of its p-Sylow subgroups. Feel free to assume that $|G|$ is divisible by exactly 2 distinct primes when you write up your solution.

(c) Let G be a finite group of order pq, where $p < q$ are distinct primes. If $q \not\equiv 1$ modulo p, show that G is cyclic.

(Keep in mind problem 6 on HW #9 and problem 6e on HW #5.)

(4) Let G be a group of order $2n$ where $n > 1$ is odd. Prove that G has a subgroup of index 2. In particular, such a G is never simple.

(*Hint:* Cayley's theorem tells us that we can view G as a subgroup of $\mathrm{Perm}(G) \cong S_{2n}$ via the map $g \mapsto \lambda_g$, where $\lambda_g : G \to G$ is the multiplication-by-g-on-the-left map. Cauchy's theorem guarantees that G has an element a of order 2. Show that λ_a is an *odd* element of S_{2n}, and use F exercise 9.29 from HW #7.)

(5) **Optional challenge problem:** Let $p < q$ be distinct primes with $q \equiv 1$ modulo p. Choose $\alpha \in \mathbb{Z}_q^\times$ satisfying $\alpha^p = 1$. (How do you know that such an α exists?)

Consider the following set:

$$A_{p,q} = \left\{ \begin{pmatrix} \alpha^k & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q) : 0 \leq k < p \right\}.$$

See problem 1 on HW #9 to recall the definition of $\mathrm{GL}_2(\mathbb{Z}_q)$.

(a) Prove that $A_{p,q}$ is a nonabelian subgroup of $\mathrm{GL}_2(\mathbb{Z}_q)$ of order pq.

(b) For $p = 2$, what group that we have studied is $A_{p,q}$ isomorphic to?
Construct the isomorphism.

(c) Find all the p-Sylow and q-Sylow subgroups of $A_{p,q}$. How many are there of each?
Feel free to do this just for $p = 3$ and $q = 7$ if you like.

(d) **Super-duper optional challenge part:** If G is a group of order pq with $q \equiv 1$ modulo p, show that G is either cyclic or isomorphic to $A_{p,q}$.