MA 541: Modern Algebra I / Fall 2019
Optional challenge problems for week #2
Due 9/19/2019 in class

(1) Let G be a group and $H \leqslant G$ a subgroup.
 (a) Show that the relation $\sim$ on G given by $a \sim b$ if and only if $a^{-1}b \in H$ for $a, b \in G$ is an equivalence relation.
 (b) Explain how the relation "congruence modulo n" on $\mathbb{Z}$ is a special case of the equivalence relation in part (a).
 (c) The relation $\sim'$ on G given by $a \sim' b$ if and only if $ab^{-1} \in H$ for $a, b \in G$ is also an equivalence relation. (Convince yourself that this is true.) Do $\sim$ and $\sim'$ give rise to the same partition of G into equivalence classes? If yes, prove this; if no, give an explicit example of a G and an H where the partitions differ.

(2) The *Gaussian integers*

$$\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}$$

is an additive subgroup of $\mathbb{C}$. Consider the relation $\equiv_{1+2i}$ on $\mathbb{Z}[i]$, where for $\alpha, \beta \in \mathbb{Z}[i]$ we say that $\alpha \equiv_{1+2i} \beta$ if there exists a $\gamma \in \mathbb{Z}[i]$ so that $\alpha - \beta = (1 + 2i)\gamma$.
 (a) Show that $\equiv_{1+2i}$ ("congruence modulo $1 + 2i$") is an equivalence relation on $\mathbb{Z}[i]$.
 (b) Let $\mathbb{Z}[i]_{1+2i}$ be the set of equivalence classes of $\mathbb{Z}[i]$ under $\equiv_{1+2i}$. Is $\mathbb{Z}[i]_{1+2i}$ a finite or an infinite set? If it is finite, how many equivalence classes are there? List or describe them all, giving explicit representatives.
 (c) Show that $\mathbb{Z}[i]_{1+2i}$ is an abelian group under addition. It is isomorphic to another group that we have studied. Explain!