MA 541: Modern Algebra I / Fall 2019
Homework assignment # 4
Due 10/1/2019

(0) Read F section 6 through Example 6.13
Read F section 13, pp. 125–129, skipping Example 13.3.

(1) Suppose G is a group, and $S \subseteq G$ a nonempty subset. Prove that S is a *subgroup* of G if and only if the following property is satisfied: given any two elements a, b in S, the element $a^{-1}b$ is also in S.

(2) Solve F exercises 6.17–6.20. Explain!

(3) Let $U := \{\alpha \in \mathbb{C} : |\alpha| = 1\}$.

(a) Show that U is a subgroup of $\mathbb{C}^\times$ (under multiplication).

(b) Prove that the map

$$\varphi : U \to \mathrm{GL}_2(\mathbb{R})$$

given by

$$a + bi \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$$

is an injective homomorphism. Here $a, b \in \mathbb{R}$.

(Note: F has a review of complex numbers in section 1.)

(4) For each item below determine whether the given map is a homomorphism of groups. Explain. If the map is a homomorphism, also give its kernel and image.

(a) The map $\varphi : \mathbb{R} \to \mathbb{Z}$ given by $\varphi(x) = \lfloor x \rfloor$. (Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

(b) The map $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$ given by $\varphi(x) = |x|$.

(c) The map $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_4$ given by $\varphi(\bar{a}) = \bar{a}$.

(d) The map $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_5$ given by $\varphi(\bar{a}) = \bar{a}$.

(e) The map $\varphi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\varphi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = a + d$.

(f) Let G be a group, and $\varphi : G \to G$ given by $\phi(g) = g^{-1}$.

(5) Show your work!

(a) Find an integer solution to the equation $27x + 43y = 1$.

(b) Find the multiplicative inverse of 58 modulo 69.

(6) Let a, b, c be integers with $\in \mathbb{Z}$ with $\gcd(a, b) = 1$. Use Bézout's lemma to show that if $a|c$ and $b|c$, then $ab|c$.

(7) Let $n \in \mathbb{Z}^+$. Prove that $\gcd(a, n)$ only depends on the residue class on a modulo n. Here $a \in \mathbb{Z}$, of course.
[In other words, show that if $a \equiv_n b$, then $\gcd(a, n) = \gcd(b, n)$.]

(8) Let G and H be groups. Prove that a group homomorphism

$$\varphi : G \to H$$

is injective if and only if $\ker \varphi = \{e_G\}$. Here e_G is the identity in G.

(9) Consider the set $\mathrm{GL}_2(\mathbb{Z}_2)$ of invertible 2×2 matrices with coefficients in $\mathbb{Z}_2$. Convince yourself that this is a group under multiplication.

(a) List the elements of $\mathrm{GL}_2(\mathbb{Z}_2)$. How many are there?

(b) Give the group table for $\mathrm{GL}_2(\mathbb{Z}_2)$.

(c) Is $\mathrm{GL}_2(\mathbb{Z}_2)$ isomorphic to another group that we have studied? Explain!