MA 541: Modern Algebra I / Fall 2019
Homework assignment #5
Due 10/8/2019

Edits 10/3/2019 in blue.

(1) (a) Give the subgroup diagram for $\mathbb{Z}_{28}$. For each subgroup $G \leq \mathbb{Z}_{28}$, list all the elements of G, and specify which of them generate G.
(b) Same question for $\mathbb{Z}_{13}^{\times}$. (Recall that $\mathbb{Z}_n^{\times}$ is the group of multiplicatively invertible residue classes modulo n under multiplication.)

(2) (a) If G is an abelian group (written multiplicatively), prove that the subset

$$G^3 := \{g^3 : g \in G\}$$

of the cubes of all the elements is a subgroup of G.
(b) Give an example of an abelian group G so that G^3 is a proper subgroup.
(c) Give an example of an abelian group G so that $G^3 = G$.

(3) Let G be a group, and $a, b \in G$ two elements with $\mathrm{ord}(a) = 5$ and $\mathrm{ord}(b) = 18$.
(a) Give three distinct elements of G with order 5. (Don't forget to explain why they are distinct!)
(b) Give three distinct elements of G of order 9 and two distinct elements of order 6.
(c) Suppose G is moreover abelian. Can you produce in G an element of 10? Of order 12? Of order 90?

(4) In F 4.19 (on homework assignment #1), you showed that the set

$$S = \mathbb{R} - \{-1\}$$

forms a group under the operation $a * b = ab + a + b$. Show that the map

$$\varphi : S \to \mathbb{R}^{\times}$$

given by $\varphi(a) = a + 1$ is a group isomorphism. (Here $\mathbb{R}^{\times}$ is, as usual, the group of nonzero real numbers under multiplication.)

(5) Let G be a group, and suppose that H and K are subgroups of G. Either prove or disprove with a counterexample each of the following.
(a) The intersection $H \cap K$ is a subgroup of G.
(b) The union $H \cup K$ is a subgroup of G.

(6) (a) Is the homomorphism $\mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4$ defined by

$$(a + 12\mathbb{Z}) \mapsto (a + 3\mathbb{Z}, a + 4\mathbb{Z})$$

an isomorphism? If is is, prove it. If it is not, specify its kernel and image.

(b) Same question for the map $\mathbb{Z}_{12} \to \mathbb{Z}_2 \times \mathbb{Z}_6$ given by $a + 12\mathbb{Z} \mapsto (a + 2\mathbb{Z}, a + 6\mathbb{Z})$.

(c) Fix positive integers m and n. What is the kernel of the homomorphism $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ given by $f(a + mn\mathbb{Z}) = (a + m\mathbb{Z}, a + n\mathbb{Z})$?

(d) **Optional challenge part:** What is the image of f from part (c)?

(e) Under what conditions on m and n is the map from part (c) an isomorphism? Formulate a precise statement and prove it.
(*Hint:* Problem (6) from HW #4 may prove useful.)

(f) **Optional challenge part:** Suppose the condition from part (e) is satisfied, so that f from part (c) is an isomorphism. Give a recipe, as precise as possible, for the inverse map.

(7) Let G be a group, and $H \leq G$ a subgroup. For any element $g \in G$, write gH for the set of products $\{gh : h \in H\}$.

(a) For $g \in G$, show that the map $H \to gH$ given by $h \mapsto gh$ is a bijection of sets.

(b) If $g \in H$, show that $gH = H$. (That is, they are the same as sets.)

(c) Show by example that gH need not be equal to H in general.

(d) **Optional challenge part:** Now suppose that G is abelian, and finite of order n. Show that $g^n = 1$ for any $g \in G$.
(*Hint:* set $H = G$ in parts (a) and (b), and compare the products of all the elements of H and of gH.)

(8) Consider the following matrices in $\mathrm{GL}_2(\mathbb{C})$:

$$\mathbf{1} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \mathbf{i} := \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad \mathbf{j} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad \mathbf{k} := \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right).$$

(a) Show that the set $Q_8 := \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ forms a subgroup of $\mathrm{GL}_2(\mathbb{C})$.

(b) Give the group table for Q_8.

(9) **Optional challenge problem (Addition in $\mathbb{Q}$):** Let $F = \mathbb{Z} \times (\mathbb{Z} - \{0\})$ be the set of rational fractions as described in class. Prove that the relation $(a, b) \sim (c, d)$ iff $ad = bc$ is an equivalence relation on F. Write $F/\sim$ for the set of equivalence classes, and prove that the binary operation $(a, b) \star (c, d) = (ad + bc, bd)$ on F determines a well-defined associative binary operation $\star$ on $F/\sim$, endowing $F/\sim$ with the structure of an abelian group.