MA 541: Modern Algebra I / Fall 2019
Homework assignment #6
Due Tuesday, October 22, at 9:30am

Minor edits 18 October 2019 in blue.
In the problems below, Q_8 refers to the group from problem (8) on HW #5.

(0) Read in F: sec. 8 through example 8.10, sec. 9 through example 9.10.

(1) Let $\sigma, \tau \in S_{15}$ be the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 13 & 2 & 15 & 14 & 10 & 6 & 12 & 3 & 4 & 1 & 7 & 9 & 5 & 11 & 8 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 14 & 9 & 10 & 2 & 12 & 6 & 5 & 11 & 15 & 3 & 8 & 7 & 4 & 1 & 13 \end{pmatrix}.$$

Write each of the following in cycle notation: $\sigma,\ \tau,\ \sigma^2,\ \sigma\tau,\ \tau\sigma$.

(2) (a) How many elements are there in S_8 with cycle structure $(5, 3)$? (Recall from class that an element with cycle structure $(5, 3)$ is a product of two disjoint cycles, a 5-cycle and a 3-cycle.) What is the order of such an element?
(b) How many elements are there in S_{15} with cycle structure $(6, 5, 4)$? What is the order of such an element?

(3) Suppose S, T, U are three sets, and $f : S \to T$ and $g : T \to U$ are two functions. Consider the function $g \circ f : S \to U$. Prove each of the following statements.
(a) If f and g are injective, then $g \circ f$ is injective.
(b) If f and g are surjective, then $g \circ f$ is surjective.
(c) If $g \circ f$ is injective, then f is injective.
(d) If $g \circ f$ is surjective, then g is surjective.

(4) Suppose every element of a group G has order dividing 2. Show that G is an abelian group.

(5) Fix $n, d \in \mathbb{Z}^+$ with $d \,|\, n$. Show that the subgroup $d\,\mathbb{Z}_n$ of $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{n/d}$. (Don't forget to check that the function that you construct giving the isomorphism is well defined!)

(6) If G is a group, define the subset

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\} \subseteq G.$$

(a) Prove that $Z(G)$ is a subgroup of G.
(b) Find $Z(G)$ for each of the following groups G:

$$\mathbb{Z},\ \mathrm{GL}_2(\mathbb{R}),\ D_3,\ D_4,\ Q_8,\ S_4.$$

(c) If $f : G \to H$ is a group homomorphism, must f map $Z(G)$ to $Z(H)$? Either prove the statement or give a counterexample.

(7) (a) Give the subgroup diagram for Q_8. Explain how you know that you've found all the subgroups.

(b) Is Q_8 isomorphic to D_4? Either construct an isomorphism or explain why no such isomorphism exists.

(c) Find as many non-isomorphic groups of size 8 as you can. Do you think you found them all?

(8) **Orders:** Let G be a group containing an element of order n for some $n \in \mathbb{Z}^+$. Suppose that $d \in \mathbb{Z}^+$ is a divisor of n. Must G contain an element of order d? If no, give a counterexample. If yes, how many elements of order d in G can you guarantee? Prove all your assertions.

(9) **LCMs:** Let a, b be in $\mathbb{Z} - \{0\}$. A *common multiple* of a and b is an integer m divisible by both a and b. The *least common multiple* of a and b (write $\operatorname{lcm}[a, b]$) is the smallest positive common multiple of a and b.

(a) We saw that $\gcd(a, b)$ is the nonnegative generator of the subgroup $a\mathbb{Z}+b\mathbb{Z}$ of $\mathbb{Z}$. Describe $\operatorname{lcm}[a, b]$ as the nonnegative generator of another "naturally occurring" subgroup of $\mathbb{Z}$ related to $a\mathbb{Z}$ and $b\mathbb{Z}$.

Now assume that both a and b are positive.

(b) If $\gcd(a, b) = 1$, prove that $\operatorname{lcm}[a, b] = ab$.

(c) Show that $\gcd(a, b) \operatorname{lcm}[a, b] = ab$.

(10) **More on orders:** Suppose that G is a group, and $g, h \in G$ are two commuting elements of finite order. Let $a = \operatorname{ord}(g)$ and $b = \operatorname{ord}(h)$.

(a) Show that the order of gh divides $\operatorname{lcm}[a, b]$.

(b) Show by example that $\operatorname{ord}(gh)$ may be strictly smaller than $\operatorname{lcm}[a, b]$.

(c) If $\gcd(a, b) = 1$, prove that $\operatorname{ord}(gh) = ab$.

(d) Prove that G always has an element of order $\operatorname{lcm}[a, b]$.

(11) **Cosets in abelian groups:** Let G be an abelian group, written additively, and $H \leq G$ a subgroup.

(a) Show that the relation $a \sim_H b$ iff $a - b \in H$ is an equivalence relation on G.

(b) For $a \in G$, write $\bar{a}$ for the equivalence class of a under $\sim_H$. Recall that $\bar{a} = \{b \in G : b \sim_H a\} \subseteq G$. Show that $\bar{a} = a + H$, where $a + H = \{a + h : h \in H\} \subseteq G$. This is a coset of H in G.

(c) For each of the following groups G and subgroups H, determine whether there are finitely many or infinitely many different cosets of H in G. If there are finitely many, list them. Otherwise, describe them geometrically.

(i) $G = \mathbb{Z}_{12}$, $H = 3\mathbb{Z}_{12}$

(ii) $G = \mathbb{R}^2$, $H = \langle(1, 2)\rangle$. In other words, $H = \{n(1, 2) : n \in \mathbb{Z}\}$. If you prefer, you may assume that $H = \{\alpha(1, 2) : \alpha \in \mathbb{R}\}$ instead.

(iii) $G = \mathbb{Q}^\times$, $H = \mathbb{Q}^+$

(iv) $G = \mathbb{C}^\times$, $H = \mathbb{R}^+$