Challenge problem added 11/6/2019. Typo in problem 2 fixed 11/7/2019.

MA 541: Modern Algebra I / Fall 2019
Homework assignment #8
Due Tuesday, 11/12/2019

(1) F exercises 10.15 and 10.16.

(2) **Indices of nested subgroups multiply.**
Solve F exercise ~~10.28~~ 10.38 to prove F Theorem 10.14.

(3) **Groups killed by 2:** Suppose G is a group with at least three elements in which every nonidentity element has order 2.
  (a) Show that G has a subgroup isomorphic to the Klein-4 group.
  (b) Show that $|G|$ is divisible by 4.

(4) (a) Show that a group of order 8 must have an element of order 2.
  (b) Show that a group of order 12 must have an element of order 2.

(5) Let H be the set of permutations in S_4 that leave the index 1 fixed.
  (a) Show that H is a subgroup of S_4. What is $(S_4 : H)$?
  (b) List the left cosets of H in S_4. What does every element of any particular left coset of H in S_4 have in common?
  (c) What does every element of a particular right coset of H in S_4 have in common?

(6) **Center of D_n:** Fix $n \geq 3$. Let $D_n = \langle r, f : r^n = f^2 = (rf)^2 = 1\rangle$ be the group of symmetries of a regular n-gon, as defined in class. Find $Z(D_n)$. It may be helpful to consider the cases of odd and even n separately.

(Here $Z(G)$ is the *center* of of a group G, the set of elements of G that commute with every element of G, defined in problem 6 of HW #6.)

(7) Let $D_6 = \langle r, f : r^6 = f^2 = rfrf = 1\rangle$ as defined in class.
  (a) Show that $H = \langle r^2, f\rangle$ is a subgroup of D_6 isomorphic to D_3.
  Give both an algebraic and a geometric explanation.
  (b) Let $\psi : D_3 \to H$ be an isomorphism from part (a). Show that the map $D_3 \times \mathbb{Z}_2 \to D_6$ given by $(d, a) \mapsto \psi(d)r^{3a}$ is a group isomorphism.
  (c) Can you generalize the statement "D_6 is isomorphic to $D_3 \times \mathbb{Z}_2$" to any D_n? (Just state whatever it is you think it true; you do not need to prove it.)
  (d) **Optional challenge problem:** Prove your statement in part (c).
  (You may prefer to wait a couple of weeks for theorems/problems on direct products and homomorphisms out of groups described via presentations.)

(8) **Images and preimages of subgroups:** Let G and H be groups, and let $f : G \to H$ be a group homomorphism.
  (a) If A is a subgroup of G, show that $f(A)$ is a subgroup of H.
  (Here $f(A)$ is the image of A under f: that is, $f(A) = \{f(a) : a \in A\}$.)
  (b) If B is a subgroup of H, show that $f^{-1}(B)$ is a subgroup of G.
  (Here $f^{-1}(B)$ is the preimage of B under f: that is $f^{-1}(B) = \{g \in G : f(g) \in B\}$.)