MA 541: Modern Algebra I / Fall 2019
Homework assignment #9
Due Tuesday, 11/19/2019

(0) Read in F: section 13, section 14, and section 15 through example 15.4.

(1) Recall that for $A = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$, or $\mathbb{Z}_n$, the group $\mathrm{GL}_2(A)$ is the set of invertible 2×2 matrices with coefficients in A and determinant in $A^\times$ (that is, the determinant has to be an invertible element of A). Moreover, $\mathrm{SL}_2(A)$ is the set of invertible 2×2 matrices with coefficients in A and determinant equal to 1.

(a) Is $\mathrm{SL}_2(\mathbb{R})$ a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$?
(b) Is the subgroup of invertible upper triangular 2×2 matrices a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$?
(c) Is $\mathrm{GL}_2(\mathbb{Z})$ a normal subgroup of $\mathrm{GL}_2(\mathbb{Q})$?
(d) Fix $N \geq 1$. Is the set

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv_N 1,\ d \equiv_N 1,\ b \equiv_N 0,\ c \equiv_N 0 \right\}$$

a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$?

In each case, explain your answer.

(2) (a) Let $H = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subseteq A_4$. Show that H is a normal subgroup of A_4. Compute the quotient A_4/H: it's isomorphic to a group we're familiar with.
(b) Compute the quotient $\mathbb{Z} \times \mathbb{Z}/\langle(1, 2)\rangle$. It is again isomorphic to a group we're familiar with. (*Hint:* Find a convenient basis for $\mathbb{Z} \times \mathbb{Z}$.)
(c) Compute the quotient $\mathbb{Z} \times \mathbb{Z}/\langle(2, 4)\rangle$. Same story.

(3) (a) Show that if $G/Z(G)$ is cyclic, then G is abelian (and hence $G = Z(G)$).
(b) Let G be a group of order pq, where p and q are prime numbers, not necessarily distinct. Show that either G is abelian or G has trivial center.

(4) **Subgroups of G/N:** Suppose G is a group and N is a normal subgroup. Let $\pi : G \twoheadrightarrow G/N$ be the natural surjective map.

(a) If H is a subgroup of G containing N, show that N is normal in H and H/N is naturally a subgroup of G/N.
(b) Conversely, show that every subgroup of G/N is of the form H/N for some subgroup H of G containing N.
(c) Show that this subgroup correspondence preserves normality: a subgroup H of G containing N is normal in G if and only if H/N is normal in G/N.

(5) **Conjugacy classes in S_n.**

(a) Suppose $\sigma = (a_1 \ \cdots \ a_k)$ is a k-cycle in S_n, and τ is also in S_n. Prove that $\tau\sigma\tau^{-1}$ is the k-cycle $\big(\tau(a_1) \ \cdots \ \tau(a_k)\big)$.

(b) Explain why conjugation by τ in S_n has the effect of replacing the indices $1, \ldots, n$ by $\tau(1), \ldots, \tau(n)$, respectively.

(c) Prove that the relation $\boxed{a \sim b \text{ if } a \text{ is conjugate to } b}$ is an equivalence relation.

The equivalence classes in G for the "is conjugate to" relation in part (c) are called *conjugacy classes*.

(d) Describe all the conjugacy classes in S_n.

(6) **Internal direct products of groups:** Suppose that G is a group and H and K are two subgroups of G satisfying the following three properties:

(a) H and K are both normal in G.

(b) $H \cap K = \{1\}$.

(c) $HK = G$.

Prove that the map $H \times K \to G$ sending (h, k) to hk is an isomorphism of groups. (*Hint:* Show that parts (a) and (b) imply that every element of H commutes with every element of K.)

(7) **Inner automorphisms:** Recall that an *automorphism* of a group G is an isomorphism $G \to G$. The set of all automorphisms of G forms a group, written $\mathrm{Aut}(G)$, under composition.

On the midterm, you showed that for every $g \in G$, the conjugation map $c_g : G \to G$ give by $x \to g^{-1}xg$ is an automorphism. Similarly, for every $g \in G$, the conjugation map $i_g : G \to G$ given by $x \to gxg^{-1}$ is an automorphism: indeed, you should check that $c_g = i_{g^{-1}}$.

(a) One of the associations $g \mapsto c_g$ or $g \mapsto i_g$ gives a natural homomorphism of groups $\alpha : G \to \mathrm{Aut}(G)$. Which one? Prove your assertion. Why doesn't the other one work?

(b) The image of α from part (a) is the subgroup $\mathrm{Inn}(G)$ of *inner automorphisms*. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

(c) What is the kernel of α?

(d) For an index i in $\{1, 2, 3, 4\}$, let H_i be the subgroup of S_4 of elements that map i to i. Find an inner automorphism of S_4 that maps H_1 to H_4.