MA 541: Modern Algebra I / Fall 2021
Homework assignment #2
Due Tuesday 9/21/21 before 5pm

Three ways to turn in your work on the due date: in class, before 5pm in the envelope hanging on MCS 127, or before 5pm emailed as an attachment to `buma541f2021@gmail.com`. If you handwrite your solutions, please try to turn in the original rather than email a scan. If you email, please have the filename identify you, this course, and the homework number.

(1) **Direct product of groups:** Let $(G, \circ)$ and $(H, *)$ be two groups with identity elements e_G and e_H, respectively. Show that $\diamond := \circ \times *$ gives a group structure on the Cartesian product $G \times H$.

Here $\diamond$ is the binary operation on $G \times H$ satisfying, for $g_1, g_2 \in G$, $h_1, h_2 \in H$,

$$(g_1, h_1) \diamond (g_2, h_2) = (g_1 \circ g_2, h_1 * h_2).$$

(2) **Groups of order 4:** Consider the following four groups of order 4:

- $\mathbb{Z}_4$, the integers modulo 4
- $\mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathrm{Symm}(▭) = \{\mathrm{id}, \mathrm{rot}(180°), \mathrm{flip}(-), \mathrm{flip}(|)\}$, the symmetries of a (nonsquare) rectangle
- $\mathrm{Rot}(\square) = \{\mathrm{id}, \mathrm{rot}(90°), \mathrm{rot}(180°), \mathrm{rot}(270°)\}$, the rotational symmetries of a square.

(a) Make a Cayley table for each group.

(b) Which of these groups have the same Cayley tables up to relabeling? Which groups are truly different? Explain!

(3) **More on groups of order 4:** Determine all possible Cayley tables for groups of order 4, up to relabeling. Explain your analysis in detail!

(If you get stuck, come ask me for a hint.)

(4) **Groups of order 6:** Do the same analysis as in problem (2) for the following groups of order 6:

- $\mathbb{Z}_6$
- $\mathbb{Z}_2 \times \mathbb{Z}_3$
- $\mathrm{Symm}(\triangle) = \{\mathrm{id}, \mathrm{rot}(120°), \mathrm{rot}(240°), \mathrm{flip}(|), \mathrm{flip}(/), \mathrm{flip}(\backslash)\}$, the symmetries of an equilateral rectangle.

(5) **William's question (challenge problem):** Prove or disprove and salvage if possible — either prove the following statement or give a counterexample, and in the latter case also try to correct the statement and prove the corrected version:

> Let $(G, \circ)$ be a set with an associative binary operation satisfying the "unique solution to linear equations property": for every $a, b \in G$, there is a unique solution in G to the equation $a \circ x = b$ and a unique solution in G to the equation $x \circ a = b$.
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> Then $(G, \circ)$ is a group.

(6) **Groups of order 5 (challenge problem):** Determine all possible Cayley tables for groups of order 5, up to relabeling. Explain!

(7) **Groups of order 6 (super-duper challenge problem):** Determine — if you dare! — all possible Cayley tables for groups of order 6.