MA 541: Modern Algebra I / Fall 2021
Homework assignment #5
Due Tuesday 10/19/21 before 5pm

Three ways to turn in your work on the due date: in class, before 5pm in the envelope hanging on MCS 127, or before 5pm emailed as an attachment to `buma541f2021@gmail.com`.

- If you handwrite your solutions, please try to turn in the original rather than emailing a scan. Please staple or otherwise connect the pages of your work. Definitely write your name on the front page.
- If you email, please have the filename identify you, the homework number, and this course, in that order.
- **Challenge problems:** Please turn solutions to challenge problems in separately. You may also turn in challenge problems later, after the deadline on the main set.

Let G be a group with identity element e. The *order* of an element g of G, denoted $|g|$ or $\mathrm{ord}(g)$, is the smallest positive integer n so that $g^n = e$, if it exists, and infinity otherwise. See also Judson 4.1.

(1) (a) Find the order of every element of $\mathbb{Z}_{12}$.
   (b) Find the order of every element of $\mathbb{Z}_{11}^{\times}$.
   (c) Find the order of every element of $\mathbb{Z} \times \mathbb{Z}_2$.

   Any observations?

(2) Let G be a group, and $a, b \in G$ two elements with $\mathrm{ord}(a) = 5$ and $\mathrm{ord}(b) = 18$.
   (a) Give three distinct elements of G with order 5. (Don't forget to explain why they are distinct!)
   (b) Give three distinct elements of G of order 9 and two distinct elements of order 6.
   (c) Suppose G is moreover abelian. Can you produce in G an element of 10? Of order 12? Of order 90?

(3) (a) Is the map $\mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4$ defined by

$$[a]_{12} \mapsto \big([a]_3, [a]_4\big)$$

   well-defined? Is it a homomorphism? Is it an isomorphism? Explain.

   (Recall that a map between groups $f : G \to H$ is a *homomorphism* if $f(ab) = f(a)f(b)$ for all $a, b \in G$, and an *isomorphism* is a bijective homomorphism.)

   (b) Same question for the map $\mathbb{Z}_{12} \to \mathbb{Z}_2 \times \mathbb{Z}_6$ given by $[a]_{12} \mapsto \big([a]_2, [a]_6\big)$.

   (c) Fix positive integers m and n. Prove that if $\gcd(m, n) = 1$, then the groups $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic.

   Problem (3) of HW #4 may be helpful.

(d) If m and n are not relatively prime, can $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ ever be isomorphic? Give an example or explain why not.

(e) **Optional challenge part:** Give a formula, as precise as possible, for the inverse of the isomorphism you constructed in (3c).

(4) (a) On HW #1 (Judson exercise 3.7) you showed that the set $S := \mathbb{R}\backslash\{-1\}$ forms an abelian group under the operation $a * b = ab + a + b$. Show that the map

$$\varphi : S \to \mathbb{R}^\times$$

given by $\varphi(a) = a + 1$ is a group isomorphism.

(b) Recall that $\mathbb{T}[m] = \{e^{2\pi ik/m} : k = 0, 1, \ldots, m-1\}$ is the subgroup of m^{th} roots of unity in $\mathbb{C}^\times$. Show that the map

$$\mathbb{Z}_m \to \mathbb{T}[m]$$

mapping $[a]$ to $e^{2\pi ia/m}$ is an isomorphism of groups.

Problem (9g) from HW #3 may be helpful.

(5) **LCMs:** Let a and b be nonzero integers. A *common multiple* of a and b is an integer m divisible by both a and b. Define the *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$, as the positive common multiple of a and b with the property that it divides any common multiple of a and b.

(a) Show that every such a, b has a unique least common multiple.

(b) We saw that $\gcd(a, b)$ is the nonnegative generator of the subgroup $\langle a, b\rangle = a\mathbb{Z} + b\mathbb{Z}$ of $\mathbb{Z}$. Describe $\operatorname{lcm}(a, b)$ as the nonnegative generator of another naturally occurring subgroup of $\mathbb{Z}$ related to $a\mathbb{Z}$ and $b\mathbb{Z}$.

Now assume that a, b are both positive.

(c) If $\gcd(a, b) = 1$, prove that $\operatorname{lcm}(a, b) = |ab|$.

(d) Show that $\gcd(a, b)\operatorname{lcm}(a, b) = |ab|$.

Problem (3) of HW #4 may be helpful again.