MA 541: Modern Algebra I / Fall 2021
Homework assignment #8
Due Tuesday 11/30/2021 by 5pm
(ok to turn in Thursday 12/2/21 in class)

Three ways to turn in your work on the due date: in class, before 5pm in the envelope hanging on MCS 127, or before 5pm emailed as an attachment to `buma541f2021@gmail.com`.

- If you handwrite your solutions, please try to turn in the original rather than emailing a scan. Please staple or otherwise connect the pages of your work. Definitely write your name on the front page. Consider using a pen rather than a pencil.
- If you email, please have the filename identify you, the homework number, and this course, in that order.
- **Challenge problems:** Please turn solutions to challenge problems in separately. You may also turn in challenge problems later, after the deadline on the main set.

Differences in (1)–(4) from the original versions posted 11/19/21 are marked in color.

(1) **Right-inverse translation action:** Let G be a group, and let $H \subseteq G$ be a subgroup.

(a) Show that $h \bullet g = gh^{-1}$ defines a (left) action of H on G. Here $h \in H$ and $g \in G$.

(b) Is this action faithful? If not, what is the kernel of this action?

(Recall that the kernel of the action of a group G on a set X is the kernel of the associated homomorphism $G \to \mathrm{Perm}(X)$. Equivalently (why?), the kernel of the action is the set $\{g \in G : g \bullet x = x \text{ for all } x \text{ in } X\}$. Equivalently (why?), the kernel of the action is the intersection of all the stabilizer subgroups.)

(c) What are the orbits of the action? Is this action transitive?

(d) What group theory fact does the orbit-stabilizer formula recover in this case?

(2) **Rotations of a cube:** Let R be the group of rotational symmetries of a cube.

(a) Consider the action of R on the set of vertices of the cube. Convince yourself that this action is transitive. How big is the orbit? Describe the stabilizer of a vertex.

(b) Consider the action of R on the set of faces of the cube. Convince yourself that this action is transitive. How big is the orbit? Describe the stabilizer of a face.

(c) Consider the action of R on the set of edges of the cube. Convince yourself that this action is transitive. How big is the orbit? Describe the stabilizer of an edge.

(d) How many elements does R have? Explain.

(e) For the purposes of this question, let's call a line that connects the midpoint of a face of the cube to the midpoint of the opposite face a "face connector". Consider the action of R on the set of face connectors of the cube. Is this action transitive? What are the orbits? Is it faithful? If not, what is the kernel of this action?

(f) **Optional challenge part:** Show that the group of all symmetries of the cube is isomorphic to $R \times \mathbb{Z}_2$. (Ask for a hint if you'd like!)

(3) **Fibers of group homomorphisms.**

(A *fiber* of a homomorphism is the preimage of a single element of the codomain.)

(a) Consider the homomorphism $f : \mathbb{R} \to \mathbb{T}$ given by $f(x) = e^{2\pi i x}$.

(i) What is $\ker f$?

(ii) What is $f^{-1}(-1)$? What is the relationship of this set to $\ker f$?

(iii) Same question for $f^{-1}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$.

(b) Let D_6 and D_3 be the dihedral groups of order 12 and 6, respectively, as usual. Set notation as follows:

$$D_6 = \langle r_6, f_6 \mid r_6^6 = f_6^2 = (r_6 f_6)^2 = 1 \rangle$$

and

$$D_3 = \langle r_3, f_3 \mid r_3^3 = f_3^2 = (r_3 f_3)^2 = 1 \rangle.$$

(i) Show that there is a homomorphism $\phi : D_6 \to D_3$ that sends r_6 to r_3 and f_6 to f_3.

(ii) What is $\ker \phi$?

(iii) For every element g of D_3, compute $\phi^{-1}(g)$. Describe this partitioning of D_6 with reference to $\ker \phi$.

(c) Let $f : G \twoheadrightarrow H$ be a surjective group homomophism with kernel K. For any $h \in H$, choose $g \in f^{-1}(h)$. Show that

$$f^{-1}(h) = gK = Kg.$$

Conclude that K is a normal subgroup of G.

(4) **Quotients of abelian groups.**

Let G be an abelian group, and let $H \subseteq G$ be a subgroup. Let aH and bH be two cosets of H in G.

(a) Show that the set of all inverses of elements of aH is a coset of H in its own right. Which coset is it?

(b) Show that the set of all pairwise products $aHbH = \{xy : x \in aH, y \in bH\}$ forms a coset of H in G in its own right. Which coset is it?

(c) Show that the set of cosets G/H forms a *group* under the operation defined in (4b). What is its identity element?

Now describe the group G/H as precisely as you can for each case below.
(Try to identify G/H as a group we've already studied.)

(d) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

(e) $G = \mathbb{R}^2$ and $H = (1, 2)\mathbb{R}$

(f) $G = \mathbb{Z}_{18}$ and $H = 3\mathbb{Z}_{18}$

(g) $G = \mathbb{R}$ and $H = \mathbb{Z}$

(5) **More quotient groups:** In each case, the quotient group is a group we're familiar with. Identify and explain!

(a) Let Z be the center of Q_8, the quaternion group (see (5bv) on HW #3).
What is Q_8/Z?

(b) Let Z be the center of D_8, the dihedral group with 16 elements (see (5a) on HW #7). What is D_8/Z?

(c) Let $V = \{e,\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\} \subseteq S_4$. Show that V is a normal subgroup of S_4. What is S_4/V?

(6) **Subgroups and quotients of G/N:** Suppose G is a group and N is a normal subgroup. Let $\pi : G \twoheadrightarrow G/N$ be the natural surjective map.

(a) If H is a subgroup of G containing N, show that N is normal in H and H/N is naturally a subgroup of G/N.

(b) Conversely, show that every subgroup of G/N is of the form H/N for some subgroup H of G containing N.

(c) Show that this subgroup correspondence preserves normality: a subgroup H of G containing N is normal in G if and only if H/N is normal in G/N.

(d) Let H be a normal subgroup of G with $N \subseteq H$. Show that the association $gN \mapsto gH$ defines a surjective group homomophism $G/N \to G/H$ with kernel H/N.

(7) Suppose that G is a group, H is a subgroup, and N is a normal subgroup.

(a) Show that $H \cap N$ is a normal subgroup of H.

(b) Show that HN is a subgroup of G containing N as a normal subgroup.

(c) Construct a natural isomorphism of groups $HN/N \to H/(H \cap N)$.
(Your work in (2) on HW #7 may be helpful.)

(8) Let p be the smallest prime dividing the order of a finite group G. Prove that any subgroup of index p in G is normal. (For $p = 2$ we already showed this in class, but that method doesn't generalize — why not?)

(*Hint:* if H has index p in G, consider the action of G on G/H by left translation. What is the kernel of this action?)

(9) (a) Suppose H and K are normal subgroups of a group G with $H \cap K = \{1\}$. Show that every element of H commutes with every element of K.

(*Hint:* Consider elements of the form $hkh^{-1}k^{-1}$ for $h \in H$ and $k \in K$.)

(b) Show the following **recognition theorem for direct products**: if H and K are two normal subgroups of a group G satisfying $H \cap K = \{1\}$ and $HK = G$, then $G \cong H \times K$. (See (4a) on HW #7, of course!)

(c) Let p be prime. Show that a group of order p^2 is isomorphic either to $\mathbb{Z}_p \times \mathbb{Z}_p$ or to $\mathbb{Z}_{p^2}$. (One possible approach: (8) and (9b).)

(10) **Optional challenge problem: Automorphisms:** Recall that an *automorphism* of a group G is an isomorphism $G \to G$. The set of all automorphisms of G forms a group, written $\mathrm{Aut}(G)$, under composition. For every $g \in G$, let $i_g : G \to G$ be the conjugation map sending x to gxg^{-1}.

(a) Show that the map $g \mapsto i_g$ gives a homomorphism of groups

$$\alpha : G \to \mathrm{Aut}(G).$$

(b) What is the kernel of α?

The image of α from (10a) is the subgroup $\mathrm{Inn}(G)$ of *inner automorphisms*.

(c) Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

The quotient group $\mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ is the group $\mathrm{Out}(G)$ of *outer automorphisms*.

(d) Give a complete description of the automorphism group of D_4 using this language. Which automorphisms are inner? What is the group $\mathrm{Out}(D_4)$?

(e) What about D_n more generally?