MA 541: Modern Algebra I / Fall 2021
Some problems that will appear on HW #8 (due 11/30/21)
Full set coming Tuesday 23 November

Fixed typo in (1a) 11/23/21.

(1) **Right-inverse translation action:** Let G be a group, and let $H \subseteq G$ a subgroup.

(a) Show that $h \cdot g = gh^{-1}$ defines a (left) action of H on G. ~~G on H.~~
Here $h \in H$ and $g \in G$.

(b) Is this action faithful? If not, what is the kernel of this action?

(Recall that the kernel of the action of a group G on a set X is the kernel of the associated homomorphism $G \to \mathrm{Perm}(X)$. Equivalently (why?), the kernel of the action is the set $\{g \in G : g \cdot x = x \text{ for all } x \text{ in } X\}$.)

(c) What are the orbits of the action? Is this action transitive?

(d) What group theory fact does the orbit-stabilizer formula recover in this case?

(2) **Rotations of a cube:** Let R be the group of rotational symmetries of a cube.

(a) Consider the action of R on the set of vertices of the cube. Convince yourself that this action is transitive. How big is the orbit? Describe the stabilizer of a vertex.

(b) Consider the action of R on the set of faces of the cube. Convince yourself that this action is transitive. How big is the orbit? Describe the stabilizer of a face.

(c) Consider the action of R on the set of edges of the cube. Convince yourself that this action is transitive. How big is the orbit? Describe the stabilizer of an edge.

(d) How many elements does R have? Explain.

(e) For the purposes of this question, let's call a line that connects the midpoint of a face of the cube to the midpoint of the opposite face a "face connector". Consider the action of R on the set of face connectors of the cube. Is this action transitive? What are the orbits? Is it faithful? If not, what is the kernel of this action?

(3) **Fibers of group homomorphisms.**

(A fiber of a homomorphism is the preimage of a single element of the codomain.)

(a) Consider the homomorphism $f : \mathbb{R} \to \mathbb{T}$ given by $f(x) = e^{2\pi i x}$.

(i) What is $\ker f$?

(ii) What is $f^{-1}(-1)$? What is the relationship of this set to $\ker f$?

(iii) Same question for $f^{-1}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i)$.

(b) Let D_6 and D_3 be the dihedral groups of order 12 and 6, respectively, as usual. Set notation as follows:

$$D_6 = \langle r_6, f_6 \mid r_6^6 = f_6^2 = (r_6 f_6)^2 = 1 \rangle;$$
$$D_3 = \langle r_3, f_3 \mid r_3^3 = f_3^2 = (r_3 f_3)^2 = 1 \rangle.$$

(i) Show that there is a homomorphism $\phi : D_6 \to D_3$ that sends r_6 to r_3 and f_6 to f_3.

(ii) What is $\ker \phi$?

(iii) For every element g of D_3, compute $\phi^{-1}(g)$. Describe this partitioning of D_6 with reference to $\ker \phi$.

(4) **Quotients of abelian groups.**

Let G be an abelian group, and let $H \subseteq G$ be a subgroup. Let aH and bH be two cosets of H in G.

(a) Show that the set of all inverses of elements of aH is a coset of H in its own right. Which coset is it?

(b) Show that the set of all pairwise products $aHbH = \{xy : x \in aH, y \in bH\}$ forms a coset of H in G in its own right. Which coset is it?

(c) Show that the set of cosets G/H forms a group under the operation defined in (4b). What is its identity element?

Now describe the group G/H as precisely as you can for each case below.
(Try to identify G/H as a group we've already studied.)

(d) $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

(e) $G = \mathbb{R}^2$ and $H = (1, 2)\mathbb{R}$

(f) $G = \mathbb{Z}_{18}$ and $H = 3\mathbb{Z}_{18}$

(g) $G = \mathbb{R}$ and $H = \mathbb{Z}$