MA 541: Modern Algebra I / Fall 2021
Some additional problems on the Sylow theorems

This set showcases some aspects of Sylow theory, including a proof of the Sylow theorems. Feel free to stop by MCS 127 next semester to chat about any of these!

First we recall the statements of the Sylow theorems. Fix a prime p. Recall that a finite p-group is a (nontrivial) group of p-power order, and a p-subgroup of a group is a subgroup that is a p-group.

Now let G be a finite group whose cardinality is divisible by p, and write $|G| = p^k m$, where m is prime to p. A subgroup P of G is called a *p-Sylow subgroup*, or sometimes *Sylow p-subgroup*, if $|P| = p^k$: that is, P is a maximal p-subgroup of G. Let $\mathrm{Syl}_p(G)$ denote the set of p-Sylow subgroups of G, and let $n_p := \left|\mathrm{Syl}_p(G)\right|$. In class we stated and used the Sylow theorems, proved by Norwegian mathematician Ludwig Sylow in 1872.

Theorem (Sylow)**.**

(I) G has p-Sylow subgroups.
Moreover, any p-subgroup of G is contained in a p-Sylow subgroup.

(II) All the p-Sylow subgroups of G are conjugate.

(III) The number n_p of p-Sylow subgroups of G satisfies

$$n_p \mid m \qquad \text{and} \qquad n_p \equiv 1 \pmod{p}.$$

Moreover, if P is any p-Sylow of G, then $n_p = [G : N(P)]$, where $N(P)$ is the normalizer of P.

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(1) **The search for simple groups:** Use the Sylow theorems and the various lemmas from class to show that the only simple groups of order n with $30 < n < 60$ are the cyclic groups of prime order.

For even more fun, do the same also for $60 < n < 168$!

The next two problems use the Sylow theorems to classify the groups of order 12.

(2) **Direct and semidirect products of Sylow subgroups.** Suppose G is a finite group with $|G| = p^k q^\ell$ for two primes p, q and $k, \ell \geq 1$. Let P be a p-Sylow subgroup of G and Q a q-Sylow subgroup of G.

(a) Show that $G = PQ$ and that $P \cap Q = \{1\}$.

(b) Show that if both P and Q are normal in G, then $G \cong P \times Q$.

How does this statement generalize to if $|G|$ is divisible by n distinct primes?

Now suppose further that P is normal in G.

(c) Show that Q acts on P by conjugation. Moreover, show that the action of each $q \in Q$ on P is an *automorphism* of P, not just a permutation, so that the action gives us a homomorphism $P \to \mathrm{Aut}(Q)$.

(d) Show that the homomorphism $P \to \mathrm{Aut}(Q)$ from (2c) determines how elements of P commute with elements of Q and hence determines the structure of G

completely. In this case we say that G is the *semidirect product* of P with Q, written $G \cong P \rtimes Q$.

(e) Show that if the homomorphism $P \to \mathrm{Aut}(Q)$ has trivial image, then Q is also normal in G (and therefore $G \cong P \times Q$).

(3) **Groups of order 12:** Let G be a group of order 12. Let P_2 be a 2-Sylow and P_3 a 3-Sylow of G.

(a) Show that n_2 is either 1 or 3, and P_2 is isomorphic either to $\mathbb{Z}_4$ or to the Klein-4 group V_4.

(b) Show that n_3 is either 1 or 4, and P_3 is isomorphic to $\mathbb{Z}_3$.

(c) Show that $n_2 = 3$ and $n_3 = 4$ cannot happen simultaneously.

(d) Use (2b) to identify G in the case that $n_2 = n_3 = 1$. These are familiar groups.

(e) Suppose $n_3 = 4$ and $P_2 \cong V_4$. Show that up to isomorphisms there is only one nontrivial map $\mathbb{Z}_3 \to \mathrm{Aut}(V_4)$. Use (2d) to identify G in this case — this is a group we have studied at a lot.

(f) Show that the case $n_3 = 4$ and $P_2 \cong \mathbb{Z}_4$ is impossible because there are no nontrivial maps $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_4)$. Use (2e).

(g) Suppose $n_2 = 3$ and $P_2 \cong V_4$. Show that up to isomorphisms there is only one nontrivial map $V_4 \to \mathrm{Aut}(\mathbb{Z}_3)$. Use (2d) to Identify G in this case — this is also a group we have studied.

(h) Finally, suppose $n_2 = 3$ and $P_2 \cong \mathbb{Z}_4$. Show that up to isomorphism there is only one nontrival map $\mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_3)$. This map determines G as in (2d). This group is new.

For what other n can you classify all groups of order n by playing similar games? For example, you can certainly do this for $n = pq$, where p and q are distinct primes — see the extra problem on the final assignment.

Finally, the next four problem will take you through the proofs of the **Sylow theorems**. You will repeatedly use the following lemma from class.

Lemma. *If G is a finite p-group acting on a finite set X, then*

$$|X| \equiv |\{\textit{fixed points of } X \textit{ under action of } G\}| \pmod{p}.$$

(4) **Normalizer subgroups:** Let G be any group and $H \subseteq G$ a subgroup.

(a) For $g \in G$, let $gHg^{-1} := \{gxg^{-1} : x \in H\}$. Show that gHg^{-1} is a subgroup of G, and that the conjugation map

$$\begin{aligned} i_g : H &\to gHg^{-1} \\ x &\mapsto gxg^{-1} \end{aligned}$$

is an isomorphism of groups.

(b) The group gHg^{-1} is a subgroup of G *conjugate* to H. Show that "is conjugate to" is an equivalence relation on subgroups of G.

(c) If $gHg^{-1} = H$ (as sets and as subgroups), then we say that g *normalizes* H. Show that the set $N(H)$ of elements $g \in G$ that normalize H is a subgroup of G containing H as a normal subgroup (that is, $H \trianglelefteq N(H)$.) This subgroup $N(H)$ is the *normalizer* of H.

(d) Show that any subgroup of G containing H as a normal subgroup is contained in $N(H)$. That is, $N(H)$ is the biggest subgroup of G containing H as a normal subgroup.

(e) Consider the action of G on its subgroups by conjugation. Show that the number of subgroups of G conjugate to H is the index $[G : N(H)]$ of the normalizer of H whenever either of these quantities is finite.

(5) **First Sylow theorem.** Recall that $k \geq 1$ is the largest power of p that divides $|G|$.

(a) First, suppose $k = 1$. Show that p-Sylow subgroups of G exist.

Now suppose $k \geq 2$. Suppose $H \subseteq G$ is a subgroup of cardinality p^r for some $1 \leq r < k$.

(b) Consider the action of H on the cosets G/H by left translation. Show that the fixed points of this action are exactly the cosets represented by $N(H)$: that is, $N(H)/H$.

(c) Prove that the index $[N(H) : H]$ is divisible by p.

(d) Prove that G contains a subgroup K that normalizes H with $(K : H) = p$. That is, we have $N(H) \supseteq K \overset{p}{\supsetneq} H$. What is the order of K?

Finally, put everything together.

(e) Use induction on k to show that any p-subgroup of G is contained in a p-Sylow subgroup of G to prove Sylow I.

(6) **Second Sylow theorem.** Let P and Q be two p-Sylows, and consider the action of P on the cosets G/Q by left translation.

(a) Suppose gQ is a fixed point of this action. Show that $P = gQg^{-1}$.

(b) Show that this action has a fixed point to prove Sylow II.

(7) **Third Sylow theorem.** Let P be a p-Sylow subgroup of G. Recall that $|G| = p^k m$, where p does not divide m.

(a) Use (4) to show that $n_p = [G : N(P)]$ and that $n_p \mid m$.

Now consider the action of P on $\mathrm{Syl}_p(G)$ by conjugation.

(b) Suppose that we can show that this action has only one fixed point. Show that this implies that $n_p \equiv 1 \pmod{p}$, the key part of Sylow III.

(c) Show that P viewed as an element of $\mathrm{Syl}_p(G)$ is a fixed point of this action.

(d) Let $Q \in \mathrm{Syl}_p(G)$ be a fixed point of this action. Show that $P \subseteq N(Q)$.

(e) Show that P and Q are conjugate by an element of $N(Q)$.
(*Hint*: Apply Sylow II to $N(Q)$.)