MA 542: Modern Algebra II / Spring 2023
Midterm start-off questions

(1) Give an example of two (or three or four) different ideals (or prime ideals or maximal ideals or principal ideals) of $A[x]$ where A is $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Z}$, $\mathbb{Z}_p$ for small primes p, or $\mathbb{Z}_p[T]/\langle \pi(t)\rangle$ for small primes p and low-degree irreducible $\pi(t)$.

(2) Let F be a field we've studied a lot. (So $F = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_p[T]/\langle \pi(t)\rangle$ as above.)

  (a) Prove that $F[x]$ is a principal ideal domain.

  (b) Find a generator of an ideal of $F[x]$ generated by three particular polynomials. (For example, find a generator of $\langle x^2, x^3 + 4x, 3x^2 + 3x\rangle$ in $\mathbb{Z}_5[x]$.)

(3) Let A be a commutative ring.
Recall that an element a of a ring A is *nilpotent* if $a^n = 0$ for some $n \geq 1$.

  (a) Prove that the nilpotent elements of A form an ideal N of A.
  This ideal is called the *nilradical* of A.

  (b) Prove that A/N has no nonzero nilpotent elements.

  (c) Prove that N is contained in every prime ideal of A.

  (d) Determine the nilradical of $\mathbb{Z}_m$ for small values of m, the nilradical of $F[x]/\langle p(x)\rangle$ for fields F and polynomials $p(x) \in F[x]$, or the nilradical of $\mathbb{Z}_m[x]$.

(4) Let F be a field, $p(x) \in F[x]$ a polynomial, and $a \in F$ an element. Show that the map

$$F[x]/\langle p(x)\rangle \to F$$

sending $[f]$ to $f(a)$ is well defined if and only if $p(a) = 0$.
Here $[f]$ is an element of $F[x]/\langle p(x)\rangle$ represented by a polynomial $f \in F[x]$.

(5) For each ring A and ideal J below, describe the elements of J. Use the fundamental isomorphism theorem for rings to describe A/J in some reasonable way. Is J prime? maximal?

  (a) $A = \mathbb{Q}[x]$ and $J = \langle x^2 + 1\rangle$

  (b) $A = \mathbb{Q}[x]$ and $J = \langle x^2 - 1\rangle$

  (c) $A = \mathbb{Q}[x]$ and $J = \{f : f(5) = 0\}$

  (d) $A = \mathbb{Z}[x]$ and $J = \langle 6\rangle$

  (e) $A = \mathbb{Z}[x]$ and $J = \langle x^2 + 1\rangle$

  (f) $A = \mathbb{Z}[x]$ and $J = \langle x^2 - 1\rangle$

  (g) $A = \mathbb{Z}[x]$ and $J = \langle 3, x^2 + 1\rangle$

(6) (a) BB 5.4.12
  (b) In the notation of BB 5.4.12, what is $Q(D_P)$?
  (c) BB 5.4.13