**MA 741: Algebra I / Fall 2020**
**Homework assignment #1**
**Due Thursday 9/17/2020**

(0) Read and review: Dummit and Foote (DF) Chapters 0–3. You may skip 3.4 at this point.

(1) **Divisible groups and $\mathbb{Q}/\mathbb{Z}$**

  (a) DF section 2.4 exercise 19
  (b) DF section 3.1 exercise 14
  (c) DF section 3.1 exercise 15

(2) **Quotients of $D_{2n}$:** DF section 3.1 exercise 34

(3) **Cauchy's theorem:** DF section 3.2 exercise 9

(4) **Universal property of quotient groups:** Let $G$ be a group and $N \subseteq G$ a normal subgroup, with $\iota : N \hookrightarrow G$ the corresponding injection. Consider the universal property satisfied by a group $Q$ defined in class:

> There is a group homomorphism $\pi : G \to Q$ so that $\pi \circ \iota : N \to Q$ is the trivial homomorphism; and if any group $X$ admits a homomorphism $f : G \to X$ with the property that $f \circ \iota : N \to X$ is the trivial map, then there is unique map $\alpha : Q \to X$ so that $f = \alpha \circ \pi$.

  (a) Show that if a group $Q$ satisfies this property, then $Q$ is unique up to unique isomorphism in the following sense: If groups $Q$ and $Q'$ both satisfy this property, with $\pi : G \to Q$ and $\pi' : G \to Q'$ the guaranteed-by-the-property maps, then there is a unique group isomorphism $\varphi : Q \to Q'$ that makes the diagram below commute.

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ Q \arrow[rr, "\varphi"'] & & Q' \end{tikzcd}$$

  (b) Show that the quotient group $G/N$ satisfies this universal property.

(5) **Universal property of products:** Let $I$ be a set of indices, and $\{G_i : i \in I\}$ a collection of groups. Let

$$G := \prod_{i \in I} G_i$$

be the direct product.

  (a) Show that $G$ satisfies the following universal property.

> For each $i \in I$ there is map $\pi_i : G \to G_i$; and given any group $X$ equipped with morphisms $f_i : X \to G_i$ for each $i \in I$, there is a unique group homomorphism $\beta : X \to G$ satisfying $f_i = \pi_i \circ \beta$ for each $i$.

  (b) Consider the following "dual" universal property of a group $F$.

> For each $i \in I$ there is a map $\iota_i : G_i \to F$, and given any group $X$ equipped with homomorphisms $f_i : G_i \to X$, there is a unique group homomorphism $\alpha : F \to X$ satisfying $f_i = \alpha \circ \iota_i$ for each $i$.

Does the direct product $G$ satisfy this second universal property? Either prove that it does or explain why not. (If you like, you may take $I$ here to be a finite set, or even just consider $I = \{1, 2\}$.)

(6) **Action of $S_n$ on $\mathbb{R}^n$:** Fix $n \geq 1$, and let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Let $\sigma \in S_n$ be a permutation, and $(a_1, \ldots, a_n) = \sum_{i=1}^n a_i e_i$ a vector in $\mathbb{R}^n$.

(a) Which of the following define a left action of $S_n$ on $\mathbb{R}^n$?

(i) $\sigma \cdot \sum_i a_i e_i = \sum_i a_i e_{\sigma(i)}$

(ii) $\sigma \cdot \sum_i a_i e_i = \sum_i a_i e_{\sigma^{-1}(i)}$

(iii) $\sigma \cdot (a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$

(iv) $\sigma \cdot (a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$

(b) Consider the items from (6a) above that define a left action of $S_n$ on $\mathbb{R}^n$. How do these actions compare?

(c) What can you say about the items from (6a) above that DO NOT define a left action of $S_n$ on $\mathbb{R}^n$?

(7) **Matrix representations of $S_3$:** A (finite-dimentional) *matrix representation* of a group $G$ over a field $K$ is a group homomorphism

$$\rho : G \to \mathrm{GL}_n(K)$$

for some $n \geq 1$. (See DF 1.4 for definitions if necessary.) We take $K = \mathbb{R}$ below.

(a) Recall that $S_3$ is isomorphic to the dihedral group $D_6$, so that we can view $S_3$ as the group of linear automorphisms of the plane preserving an equilateral triangle centered at the origin. To fix ideas, let the triangle have vertices $(1, 0)$, $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ in $\mathbb{R}^2$.

Express this action as an explicit matrix representation $\rho : S_3 \to \mathrm{GL}_2(\mathbb{R})$.

(b) The action of $S_3$ on $\mathbb{R}^3$ suggested in problem (6) is called the *permutation representation* of $S_3$. Construct this representation

$$\pi : S_3 \to \mathrm{GL}_3(\mathbb{R})$$

explicitly in terms of matrices.

(c) Show that the line $\ell$ spanned by $v = (1, 1, 1) \in \mathbb{R}^3$ is *stable* under the action of $S_3$ given by $\pi$ (that is, if $x \in \ell$ and $\sigma \in S_3$, then $\sigma x \in \ell$ as well).

(d) Find a complement $P$ to the line $\ell$ (i.e., a plane $P$ in $\mathbb{R}^3$ containing the origin but not containing $\ell$) that is also stable under the action of $S_3$ given by $\pi$.

(e) Choosing a convenient basis for $P$, express the action of $S_3$ given by $\pi$ on $P$ as an explicit matrix representation

$$\sigma : S_3 \to \mathrm{GL}_2(\mathbb{R}).$$

(f) You now have two dimension-2 matrix representations of $S_3$: $\rho$ from (7a) and $\sigma$ from (7e). Compare them!