MA 741: Algebra I / Fall 2020
Homework assignment #2
Due Thursday 9/24/2020

(0) Read and review: DF sections 1.7, 2.2, 4.1, 4.2, 4.3. 5.1.

Also recall the following theorem from class: if p is a prime and G is a finite p-group acting on a finite set X, then

$$|X| \equiv |\{\text{fixed points of } X\}| \pmod{p}.$$

(1) **McKay's proof of Cauchy's theorem:** Let G be a finite group and p a prime number dividing $|G|$. Prove that G has an element of order p as follows.

Let G^p be the product of p copies of G, and $H \subset G^p$ the subset of p-tuples $(g_1, \cdots, g_p)$ whose product $g_1 \cdots g_p$ is 1. Convince yourself that H is a subgroup of G^p.

(a) Show that $\mathbb{Z}/p\mathbb{Z}$ acts on both G^p and H by cyclic permutation.
(b) Show that $(g_1, \cdots, g_p) \in G^p$ is a fixed point of the action if and only if

$$g_1 = g_2 = \cdots = g_p.$$

(c) Show that a fixed point of the action of $\mathbb{Z}/p\mathbb{Z}$ on H corresponds to an element $g \in G$ of order dividing p.
(d) Use the action of $\mathbb{Z}/p\mathbb{Z}$ on H to prove that

$$|H| \equiv |\{h \in H : h \text{ is fixed by the action of } \mathbb{Z}/p\mathbb{Z}\}| \pmod{p}.$$

(e) Prove Cauchy's theorem: if G is a finite group and p divides $|G|$, then G has an element of order p.

(2) **Sylow theorems:** Let p be a prime dividing the cardinality n of a finite group G, and write $n = p^k m$, where m is prime to p. A subgroup P of G is called a *p-Sylow subgroup*, or sometimes *Sylow p-subgroup*, if $|P| = p^k$: that is, P is of maximal p-power order for any subgroup of G.

Your main goal is to prove the first Sylow theorem: If p divides $|G|$, then G has at least one p-Sylow subgroup.

First, let G be any group, and consider the action of G on the set of its subgroups by conjugation. If $H \subseteq G$ is any subgroup, then the stabilizer $\mathrm{Stab}(H)$ under this action is called the *normalizer* $N_G(H)$, and any element in $N_G(H)$ *normalizes* H.

(a) Show that $N_G(H)$ is a subgroup of G containing H with the property that H is normal in $N_G(H)$. (In fact $N_G(H)$ is the largest subgroup of G in which H is normal. Why?)

Now back to the case that p divides $|G|$. Let $k > 0$ be the largest power of p dividing $|G|$. Suppose that we have found a subgroup P of G of cardinality p^r for some r with $1 \leq r < k$.

(b) Consider the action of P on the cosets G/P by left translation. What are the fixed points of the action?

(c) Prove that the index $(N_G(P) : P)$ is divisible by p.

(d) Prove that G contains a subgroup Q that normalizes P with $(Q : P) = p$: that is:

$$N_G(P) \supseteq Q \overset{p}{\supsetneq} P.$$

What is the order of Q?

(e) Finally use induction on the largest power of p dividing $|G|$ to prove the First Sylow Theorem.

(f) Now prove the Second Sylow Theorem: all the p-Sylow subgroups of G are conjugate. Let P and Q be two p-Sylows, and consider the action of P on the cosets G/Q by left translation. Show that there's fixed point of the action. Conclude that if coset gQ is a fixed point, then $P = gQg^{-1}$.

(3) **Exactness:** A sequence of groups and homomorphisms

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} G_n$$

is called *exact at* G_i for some i with $1 \leq i \leq n-1$ if $\ker f_i = \operatorname{im} f_{i-1}$ as subgroups of G_i. The sequence is called *exact* if it is exact at i for every $i = 1, 2, \ldots, n-1$.

We can also consider exactness of sequences that are infinite: on the left, or the right, or on both sides (bi-infinite).

Let

$$(*) \qquad \cdots \longrightarrow G_{-2} \xrightarrow{f_{-2}} G_{-1} \longrightarrow \underset{\displaystyle G_0}{\overset{\displaystyle 1}{\|}} \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_{-1}} G_3 \longrightarrow \cdots$$

be a (possibly infinite) sequence of groups connected by group homomorphisms. Note that $G_0 = 1$, the trivial group.

(a) Show that sequence $(*)$ is exact at $G_0 = 1$.

(b) Show that $(*)$ is exact at G_{-1} if and only if f_{-2} is surjective.

(c) Show that $(*)$ is exact at G_1 if and only if f_1 is injective.

(d) Show that $(*)$ is exact if and only if both of the following are exact:

$$\cdots \longrightarrow G_{-2} \xrightarrow{f_{-2}} G_{-1} \longrightarrow 1$$

and

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_{-1}} G_3 \longrightarrow \cdots.$$

A *short exact sequence* of groups is an exact sequence of the form

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1.$$

(e) Let

$$1 \longrightarrow G \xrightarrow{f} H \xrightarrow{g} K \longrightarrow 1$$

be a sequence of groups connected by group homomorphisms. Show that this sequence is short exact if and only if all three of the following hold:

(i) f is injective;

(ii) g is surjective;

(iii) g induces an isomorphism $H/f(G) \xrightarrow{\sim} K$.

(4) **Automorphisms:** An *automorphism* of a group G is an isomorphism $G \longrightarrow G$. Write $\operatorname{Aut}(G)$ for the set of automorphisms of a group G.

(a) Show that $\operatorname{Aut}(G)$ forms a group under composition.

Now let N be a normal subgroup of a group G.

(b) Show that the action of G on N by conjugation induces a group homomorphism
$$\pi_N : G \longrightarrow \operatorname{Aut}(N).$$
(That is, for $g \in G$ we let $\pi(g)$ be the map $N \longrightarrow N$ given by $n \mapsto gng^{-1}$.)

(c) What is the kernel of π_N?

In the case that $N = G$, elements in the image of π_G are called *inner automorphisms* of G. The subgroup of inner automorphisms is sometimes denoted $\operatorname{Inn}(G)$.

(d) Show that $\operatorname{Inn}(G)$ is a normal subgroup of $\operatorname{Aut}(G)$.

The quotient $\operatorname{Aut}(G)/\operatorname{Inn}(G)$ is the group $\operatorname{Out}(G)$ of *outer automorphisms.*

(e) Show that we have an exact sequence
$$1 \longrightarrow Z(G) \longrightarrow G \xrightarrow{\pi_G} \operatorname{Aut}(G) \longrightarrow \operatorname{Out}(G) \longrightarrow 1.$$

(f) What does the exact sequence in (4e) look like for D_8, the symmetry group of a square? Justify your assertions.

(5) **Abelianization:** Let G be a group, and a, b elements of G. The element $aba^{-1}b^{-1}$ is called the *commutator* of a and b, often denoted $[a, b]$. Let $[G, G]$ be the subgroup of G generated by all the commutators of all the pairs of elements of G.

(a) Show that $[G, G]$ is a normal subgroup of G.

(b) Show that the quotient $G/[G, G]$ is an abelian group, called the *abelianization* of G, and often denoted G^{ab}.

(c) Let N be a normal subgroup of G. Show that the quotient G/N is abelian if and only if N contains $[G, G]$.

(d) Show that G^{ab} satisfies the following (initial) universal property: it's an abelian group equipped with a map $\pi : G \longrightarrow G^{\mathrm{ab}}$, and any group homomorphism $f : G \longrightarrow A$ to an abelian group A factors through π: that is, there is a unique map $\alpha : G^{\mathrm{ab}} \longrightarrow A$ so that $f = \alpha \circ \pi$.

Now let $f : G \longrightarrow H$ be a group homomorphism.

(e) Show that f induces a homomorphism of abelian groups
$$f^{\mathrm{ab}} : G^{\mathrm{ab}} \longrightarrow H^{\mathrm{ab}}.$$

(f) Must f^{ab} be injective if f is injective? Prove or give a countexample.

(g) Must f^{ab} be surjective if f is surjective? Prove or give a countexample.

(h) More generally, suppose $1 \longrightarrow G \xrightarrow{f} H \xrightarrow{g} K \longrightarrow 1$ is an exact sequence of groups. By part (5e) above, we obtain a sequence of maps
$$1 \longrightarrow G^{\mathrm{ab}} \xrightarrow{f^{\mathrm{ab}}} H^{\mathrm{ab}} \xrightarrow{g^{\mathrm{ab}}} K^{\mathrm{ab}} \longrightarrow 1.$$
At which of G^{ab}, H^{ab}, K^{ab} is this sequence exact? Prove your assertions.

(6) **Torsion in abelian groups:** Let A be an abelian group, written additively. Fix $n \geq 1$. An element $a \in A$ is an *n-torsion element* if $na = 0$. Let

$$A[n] := \{a \in A : na = 0\}$$

be the subset of n-torsion elements, and let

$$T(A) := \bigcup_{n \geq 1} A[n],$$

the set of all *torsion* elements of A. If $T(A) = \{0\}$ then A is said to be *torsion free.*

(a) Show that both $A[n]$ and $T(A)$ are subgroups of A.
(b) Show that $A/T(A)$ is torsion free.
(c) Let A be a free abelian group. Show that A is torsion free.
(d) Show that $\mathbb{Q}$ is torsion free but not free as an abelian group.

(7) **Optional challenge problems:** The following are all implied by the structure theorem for finitely generated abelian groups. Can you show show these without appealing to that theorem, or the notion of noetherianness?

(a) If A is a torsion-free finitely generated abelian group, then A is free.
(b) If A is finitely generated free abelian group, and $B \subset A$ is a subgroup, then B is also a finitely generated free abelian group, and $\mathrm{rank} B \leq \mathrm{rank} A$. (In fact, it's true without the finitely generated condition: a subgroup of a free abelian group is free abelian.)
(c) If A is a finitely generated abelian group, then $T(A)$ is a finitely generated (equivalently, finite) group.