MA 741: Algebra I / Fall 2020
Homework assignment #3
Due Thursday, October 1, 2020

When you turn in your solutions, please do not forget to identify yourself, this course, and the number of the HW set in the document name.

(0) Read and review in DF...

  (a) Section 5.2.
  (b) Section 6.3, including pp. 218–220 on presentations.
  (c) Smith normal form and the proof of Theorem 3 of section 5.2: see pp. 470–471, where you can take R to be $\mathbb{Z}$.

(1) DF section 5.2 exercises 2 and 3.

(2) **Uniqueness in the structure theorem for finitely generated abelian groups:** Let p be a prime number, and $(a_1, a_2, \ldots, a_k)$ be a *partition* of the positive integer α (that is: $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ and $a_1 + \cdots + a_k = \alpha$), and let

$$A := \mathbb{Z}/p^{a_1}\mathbb{Z} \times \cdots \mathbb{Z}/p^{a_k}\mathbb{Z}$$

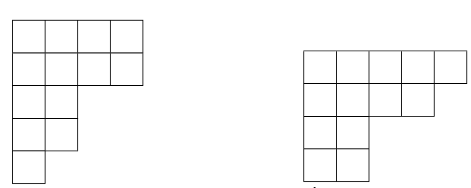
be the abelian group of order p^α.

  (a) What is the exponent of A?

  (The *exponent* of a group G is the least positive integer n so that $g^n = 1$ for all $g \in G$, if such an n exists, and infinity otherwise. If G is finite abelian, the exponent of G is the least $n \geq 1$ such that $G = G[n]$.)

  (b) What is the cardinality of $A[p]$? Of $A[p^2]$? Find an expression for the cardinality $A[p^n]$ for any $n \geq 1$. Explain.

  (The following ideas might be helpful. As defined in class, a *partition* of an integer α is a tuple $a = (a_1, \ldots, a_k)$ with $a_1 + \cdots + a_k = \alpha$ ordered so that $a_1 \geq \cdots \geq a_k \geq 1$. If $a = (a_1, \ldots, a_k)$ partition of α, we can consider the *conjugate partition* a' of α, defined as follows: the i^{th} part of a' is the number of *parts* a_j of a that are at least i. A helpful way to visualize this is through so-called Ferrers diagrams:



$a = (4, 4, 2, 2, 1)$ $\qquad$ $a' = (5, 4, 2, 2)$

The diagram of the conjugate partition a' is obtained from the diagram for a by flipping it over the diagonal, as in the example above.)

(c) Now suppose that $(b_1, b_2, \dots, b_\ell)$ is also a partition of α, and let

$$B := \mathbb{Z}/p^{b_1}\mathbb{Z} \times \cdots \mathbb{Z}/p^{b_\ell}\mathbb{Z}.$$

Prove that $A \cong B$ if and only if $(a_1, \dots, a_k) = (b_1, \dots, b_\ell)$ as partitions of α (that is, if and only if $k = \ell$ and $a_i = b_i$ for all $i = 1, \dots, k$).

(d) Prove that if T is a finite abelian group, then the elementary divisor decomposition of T (Theorem 5 of DF 5.2) is unique.

(e) Use (2d) to conclude that if T is a finite abelian group, then the invariant factor decomposition of T (Theorem 3 of DF 5.2) is unique.

(3) **Splitting of exact sequences:** Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of abelian groups. Recall that $s : C \longrightarrow B$ is a *section* of β if it is a right inverse to β: that is, $\beta \circ s = \mathrm{id}_C$.

(a) Show that the following are equivalent:
  (i) β has a section;
  (ii) α has a left inverse: a map $t : B \longrightarrow A$ such that $t \circ \alpha = \mathrm{id}_A$;
  (iii) there is an isomorphism $B \cong A \oplus C$ under which α is the natural inclusion $A \hookrightarrow A \oplus C$ and β is the projection $A \oplus C \twoheadrightarrow C$.

If the equivalent conditions above are satisfied, then the sequence is said to (be) *split*.

(b) Which of the following exact sequences of abelian groups are split? Explain.
  (i) The sequence

$$1 \longrightarrow \mathbb{Q}^+ \longrightarrow \mathbb{Q}^\times \longrightarrow \{\pm 1\} \longrightarrow 1,$$

  where the first map is the inclusion and the second sends $\alpha \in \mathbb{Q}$ to $\alpha/|\alpha|$.
  (ii) The sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow \mathbb{S}^1 \longrightarrow 1,$$

  where $\mathbb{S}^1 \subset \mathbb{C}^\times$ is the unit circle. Here the first map sends $x \in \mathbb{R}$ to e^x and the second map sends $\alpha \in \mathbb{C}^\times$ to $\alpha/|\alpha|$.
  (iii) The sequence

$$1 \longrightarrow \mathbb{Q}^+ \xrightarrow{2} \mathbb{Q}^+ \longrightarrow A \longrightarrow 1,$$

  where the first map is squaring and A is the cokernel.
  (iv) The sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

  (v) For a finitely generated abelian group A, the sequence

$$0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0.$$

  Here $T(A)$ is the torsion subgroup of A: see (6) on HW #2.

(vi) The sequence

$$0 \longrightarrow \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \longrightarrow \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \longrightarrow A \longrightarrow 0,$$

where A is the cokernel of the first map.

Hint: Which elements of A are divisible by every $n \geq 1$?
Same question for $\prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$.

(vii) For any abelian group A, the sequence

$$0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0.$$

(4) **Recognition theorem for direct products**

(a) Let G be a group with normal subgroups $H_1, \ldots, H_n$ so that the following two conditions are satisfied

(i) $G = H_1 H_2 \cdots H_n$ and

(ii) for each $1 \leq i \leq n$, we have

$$H_i \cap H_1 \cdots H_{i-1} H_{i+1} \cdots H_n = \{1\}.$$

Show that the map

$$H_1 \times \cdots \times H_n \longrightarrow G$$

given by

$$(h_1, \ldots, h_n) \mapsto h_1 \cdots h_n$$

is an isomorphism of groups.

(b) Let A be a finite abelian group with $p_1, \ldots, p_k$ the distinct primes dividing $|A|$. Show that A factors as a product of its p_i-Sylow subgroups. (See (2) on HW #2 for definitions and basic facts about Sylow subgroups.)

(5) A *character* of a group G to a field K is a one-dimensional (matrix) representation of G over K (see (7) on HW #1): that is a group homomorphism

$$\chi : G \longrightarrow \mathrm{GL}_1(K) = K^\times.$$

Below we assume that $K = \mathbb{C}$.

(a) Find all the complex characters of $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and S_3.

(b) Show that the set of all complex characters of a group G forms an abelian group under pointwise multiplication of their values. This group is often denoted $\hat{G}$.

(c) If G is a finite abelian group, prove that $G \cong \hat{G}$. (See DF exercise 14 on page 167, where $\hat{G}$ is called the *dual group* to G, for some ideas.) Note that there are many such isomorphisms.

(d) If $f : G \longrightarrow H$ is a homomorphism between groups, show that precomposing with f yields a homomorphism $\hat{f} : \hat{H} \longrightarrow \hat{G}$. If f is injective, must $\hat{f}$ be injective? surjective? What if f is surjective?

(6) Find by hand the Smith normal form of the matrix

$$M = \begin{pmatrix} 20 & 40 & 20 \\ 36 & 84 & 40 \\ -66 & -112 & -60 \end{pmatrix}.$$

Explain.
What are the elementary divisors of the cokernel of the map $\mathbb{Z}^3 \longrightarrow \mathbb{Z}^3$ given by M?

(7) (a) Let S be a set, $F(S)$ the free group with generating set S, and $F^{\mathrm{ab}}(S)$ the free abelian group with basis S. Show that the abelianization of $F(S)$ satisfies the universal property satisfied by $F^{\mathrm{ab}}(S)$. Conclude that $F(S)^{\mathrm{ab}}$ is naturally isomorphic to $F^{\mathrm{ab}}(S)$.

(b) DF section 6.3 exercise 1.