MA 741: Algebra I / Fall 2020
Some problems that will appear on homework assignment #3 or later

(1) DF section 5.2 exercises 2 and 3.

(2) DF section 5.2 exercise 13. You're being asked to show that some group satisfies a universal mapping property. Can you recast this in the language we have been using in class?

(3) **Uniqueness in the structure theorem for finitely generated abelian groups:** Let p be a prime number, and $(a_1, a_2, \dots, a_k)$ be a *partition* of the positive integer α (that is: $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ and $a_1 + \cdots + a_k = \alpha$), and let

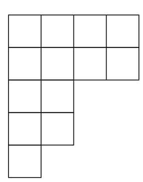
$$A := \mathbb{Z}/p^{a_1}\mathbb{Z} \times \cdots \mathbb{Z}/p^{a_k}\mathbb{Z}$$

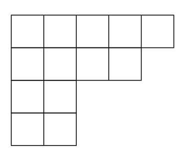
be the abelian group of order p^α.

(a) What is the exponent of A? (The *exponent* of a group G is the least positive integer n so that $g^n = 1$ for all $g \in G$, if such an n exists, and infinity otherwise. If G is finite abelian, the exponent is the least $n \geq 1$ such that $G = G[n]$.)

(b) What is the cardinality of $A[p]$? Of $A[p^2]$? Give an expression for the cardinality $A[p^n]$ for any $n \geq 1$.

The following ideas and terms might prove helpful: given $a = (a_1, \dots, a_k)$ is a partition of α, we can consider the *conjugate partition* a' of α, defined as follows: the i^{th} part of a' is the number of parts of a that are at least i. A helpful way to visualize this is through Ferrers diagrams: the diagram of the conjugate partition a' is obtained from the diagram for a by flipping it over the diagonal, as in the example below.


$a = (4, 4, 2, 2, 1)$


$a' = (5, 4, 2, 2)$

(c) Now suppose that $(b_1, b_2, \dots, b_\ell)$ is also a partition of α, and let

$$B := \mathbb{Z}/p^{b_1}\mathbb{Z} \times \cdots \mathbb{Z}/p^{b_\ell}\mathbb{Z}.$$

Prove that $A \cong B$ if and only if $(a_1, \dots, a_k) = (b_1, \dots, b_\ell)$ as partitions of α: that is, if and only if $k = \ell$ and $a_i = b_i$ for all $i = 1, \dots, k$.

(d) Prove that if T is a finite abelian group, then the elementary divisor decomposition of T (Theorem 5 of DF 5.2) is unique.

(e) Use (3d) to conclude that if T is a finite abelian group, then the invariant factor decomposition of T (Theorem 3 of DF 5.2) is unique.

(4) A *character* of a group G to a field K is a one-dimensional (matrix) representation of G over K (see HW #1 (**??**)): that is a group homomorphism

$$\chi : G \longrightarrow \mathrm{GL}_1(K) = K^\times.$$

Below we assume that $K = \mathbb{C}$.

(a) Find all the complex characters of $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and S_3.
(b) Show that the set of all complex characters of a group G from an abelian group under pointwise multiplication of their values. This group is often denoted $\hat{G}$.
(c) If G is a finite abelian group, prove that $G \cong \hat{G}$. (See DF exercise 14 on page 167, where $\hat{G}$ is called the *dual group* to G, for some ideas.)
(d) ...contravariance
(e) ...double dual

(5) ...Splitting of exact sequences of abelian groups

(6) ...Semidirect products

(7) ...Recognition theorem for finite direct products. Finite abelian group factors as product of p-Sylows.

(8) ...Smith normal form computation