MA 741: Algebra I / Fall 2020
Homework assignment #5
Due weekend of October 24-25, 2020

Clean copy.
To turn in your work, please email your well-titled document (title should identify you, this course, and the HW set number) to `buma741fall2020@gmail.com` with "HW 5" in the subject line. Please indicate whom you worked with on the problem set.

(0) Suggested reading: DF 7.4, 8.1, 8.2, 9.1, 9.2; Aluffi III.1–4, V.1–3; Atiyah-Macdonald chapter 1. Note that all rings in Atiyah-Macdonald (AM) are commutative.

An element a of a ring A is called *nilpotent* if $a^n = 0$ for some $n \geq 1$.

(1) (a) Show that the set of all nilpotent elements of a commutative ring A forms an ideal of A. What if A is not commutative?
(b) Let a be a nilpotent element in a commutative ring A. Show that $1 + a$ is a unit of A. What if A is not commutative?
(c) Deduce that the sum of a nilpotent element and a unit in a commutative ring A is a unit of A. What if A is not commutative?

Part of this question comes from AM exercise 1.1.

(2) Let A be a commutative ring, and $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that
(a) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.
(b) f is nilpotent $\iff a_0, \ldots a_n$ are all nilpotent.
(c) f is a zero divisor $\iff$ there exists $a \neq 0$ in A with $af = 0$.
(d) f is said to be *primitive* if $(a_0, \ldots, a_n) = 1$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff f$ and g are primitive.
(*Hint:* If fg is not primitive, let $\mathfrak{m}$ be a maximal ideal containing the coefficients of fg. Now either work in $A/\mathfrak{m}[x]$ or show that various coefficients are in $\mathfrak{m}$ by hand.)
(e) Show that $A[x]$ is an integral domain if and only if A is an integral domain.

Parts (a)–(d) is AM exercise 1.2. See that prompt for more hints!

(3) Let A be a commutative ring, and let $A[\![x]\!]$ be the ring of *formal power series* $f = \sum_{n=0}^{\infty} a_nx^n$. (These power series are called formal because we are not concerned with convergence.)
(a) Show that f is a unit in $A[\![x]\!] \iff a_0$ is a unit in A.
(b) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?

This is part of AM exercise 1.5.

(4) **Nilradical ideal:** The ideal of all nilpotent elements of a commutative ring A is called the *nilradical* of A. Call it $\mathfrak{N}$.
(It is sometimes also denoted $\sqrt{0}$ or $r(0)$ as it is the *radical* of the zero ideal. More generally, the *radical* of any ideal $\mathfrak{a}$ is $r(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{a \in A : a^n \in \mathfrak{a} \text{ for some } n \geq 1\}$.)

(a) Show that $A/\mathfrak{N}$ has no nilpotent elements (such a ring is called *reduced*).
(b) If $a \in A$ is nilpotent, show that a is in every prime ideal of A.
(c) The converse is also true: an element in every prime ideal of A is nilpotent. For the proof, you will need to use Zorn's lemma (see Aluffi V.3, for example). Suggested steps: Take any nonnilpotent element a of A. Let Σ be the set of ideals $\mathfrak{a}$ of A with the property that no positive power of a is in $\mathfrak{a}$, ordered by inclusion.
  (i) Show that Σ is nonempty.
  (ii) Show that every chain in Σ has an upper bound.

  Zorn's lemma now implies that Σ has a maximal element. Call it $\mathfrak{m}$. Show that $\mathfrak{m}$ is a prime ideal as follows.
  (iii) Prove that for any $x \in A$ we have $x \notin \mathfrak{m}$ if and only if there exists a positive integer n with $a^n \in \mathfrak{m} + (x)$.
  (iv) Show that $x, y \notin \mathfrak{m}$ implies that $xy \notin \mathfrak{m}$. Conclude that $\mathfrak{m}$ is prime.

In other words, the nilradical of A is the intersection of all the prime ideals of A.

(d) If $\mathfrak{a}$ is an ideal of A, show that the radical ideal $\sqrt{\mathfrak{a}}$ is the intersection of all the prime ideals of A containing $\mathfrak{a}$. (*Hint:* Work in $A/\mathfrak{a}$.)

(5) **Zariski topology on** $\operatorname{Spec} A$ **(AM exercises 1.15, 1.16):** Let A be a commutative ring, and X the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A that contain E. Show each of the following.

(a) If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\sqrt{\mathfrak{a}})$.
(b) $V(0) = X$ and $V(1) = \varnothing$
(c) If $\{E_i\}_{i \in I}$ is any collection of subsets of A, then $V\left(\cup_{i \in I} E_i\right) = \cap_{i \in I} V(E_i)$.
(d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

  Proposition 1.11 in AM may be helpful.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski* topology. The topological space X is called the *(prime) spectrum* of A, written $\operatorname{Spec} A$.

(e) Think about and draw $\operatorname{Spec} \mathbb{Z}$, $\operatorname{Spec} \mathbb{C}$, $\operatorname{Spec} \mathbb{C}[x]$, $\operatorname{Spec} \mathbb{Z}[x]$.

(6) Read about the ring of *Gaussian integers* $\mathbb{Z}[i]$ being a Euclidean domain with respect to the norm $N(a+bi) = a^2 + b^2$: Aluffi V.6 has a lovely geometric treatment; or see DF example 3 starting on p. 271 (or Aluffi exercise V.6.12) for an algebraic proof.

(a) View the ring $\mathbb{Z}[\sqrt{-2}]$ as a subring of $\mathbb{C}$, and define the *norm* of $\alpha = a + b\sqrt{-2}$ as $N(\alpha) := a^2 + 2b^2$. (Note that the norm is the square of the complex absolute value.) Show that the ring $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with respect to the norm: that is, show that for all $\alpha, \beta \in \mathbb{Z}[\sqrt{-2}]$ with $\beta \neq 0$, there exists $q, r \in \mathbb{Z}[\sqrt{-2}]$, with $0 \leq N(r) < N(\beta)$, satisfying $\alpha = \beta q + r$. You may use either a geometric or an algebraic argument (or both!).

  It follows that $\mathbb{Z}[\sqrt{-2}]$ is a PID and a UFD.

(b) What happens when you try to run the same argument for $\mathbb{Z}[\sqrt{-3}]$? Here use the norm $N(a + b\sqrt{-3}) = a^2 + 3b^2$.

(c) Explain why the equation

$$2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$$

shows that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

(d) Do you still have these problems if you replace $\mathbb{Z}[\sqrt{-3}]$ by the larger ring

$$\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] = \left\{\frac{a + b\sqrt{-3}}{2} : a \equiv b \bmod 2 \text{ in } \mathbb{Z}\right\}?$$

Note that $N(\alpha)$ is still integral for $\alpha \in \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$.

Come talk to me about this problem if you get stuck!

(7) **Optional challenge problem:** Keep reading about the two-square theorem of Fermat (Aluffi V.6.3). How do odd integer primes of $\mathbb{Z}$ factor in $\mathbb{Z}[\sqrt{-2}]$? Factor all the primes up to 20 in $\mathbb{Z}[\sqrt{-2}]$ as products of irreducibles.