MA 741: Algebra I / Fall 2020
Homework assignment #7
Due week of November 20, 2020

To turn in your work, please email your well-titled document (title should identify you, this course, and the HW set number) to `buma741fall2020@gmail.com` with "HW 7" in the subject line. Please indicate with whom you worked on the problem set.

(1) **Exactness of Hom-functors:** Let R and S be rings. A covariant additive functor $\mathcal{F} : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ is called *left exact* if given an exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

in R-**mod**, the sequence

$$0 \to \mathcal{F}(A) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}(B) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(C)$$

is exact in S-**mod**. Similarly, such an $\mathcal{F}$ is *right exact* if if given exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

in R-**mod**, the sequence

$$\mathcal{F}(A) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}(B) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(C) \to 0$$

is exact in S-**mod**. Finally $\mathcal{F}$ is *exact* if it is both left exact and right exact (equivalently, it transforms short exact sequences to short exact sequences).

(a) For an R-module X, consider the functor $h_X = \mathrm{Hom}_R(X, -)$ from R-modules to abelian groups. Show that h_X is left exact.

(b) Furthermore, show that a sequence

$$0 \to M \to N \to P$$

of R-modules is exact if and only if

$$0 \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, N) \to \mathrm{Hom}_R(X, P)$$

is exact for every R-module X.

(c) Now consider the functor $h^X = \mathrm{Hom}_R(-, X)$, which is covariant as a functor from $R\text{-}\mathbf{mod}^{\mathrm{op}}$ to $\mathbf{Ab}$. Show that h^X is also left exact. What does this mean here?

(d) Show that a sequence

$$M \to N \to P \to 0$$

of R-modules is exact if and only if

$$0 \to \mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R(N, X) \to \mathrm{Hom}_R(M, X)$$

is exact for every R-module X.

(2) **Direct limits.**

(a) Read Atiyah-Macdonald exercise 14 on pp. 32–33. There is nothing to show. Note that the setup is equivalent to taking a (directed) poset category I and considering a functor $\mathcal{F}$ from I to A-modules, with $M_i = \mathcal{F}(i)$.

(b) Atiyah-Macdonald exercise 15 on p. 33.

(c) Atiyah-Macdonald exercise 16 on p. 33.

(d) Consider the poset $\mathbb{Z}^+$ with the "divides" relation. Show that this poset is directed. For each n, consider the abelian group $\mathbb{Z}/n\mathbb{Z}$; whenever $n \mid m$, let $\mu_{n,m} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the map sending 1 to $\frac{m}{n}$. What is $\varinjlim \mathbb{Z}/n\mathbb{Z}$?

Hint: Think of $\mathbb{Z}/n\mathbb{Z}$ as $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

(3) Some tensor products.

(a) Compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ for $m, n \geq 1$.
(b) Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.
(c) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} = \mathbb{Q}$.
(d) Let G be a finitely generated abelian group, and p a prime. Describe $G \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ in terms of the elementary divisor decomposition of G.
(e) Let G be a finite abelian group, and p^k the largest power of a prime p dividing $|G|$. What is $G \otimes_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}$?

(4) **Group representations:** A *representation* of a group G on a vector space V over a field K is an action of G on V by K-linear transformations. In other words, a representation of G over K is a pair (ρ, V), where V is a K-vector space and

$$\rho : G \to \mathrm{GL}_K(V)$$

is a group homomorphism. Sometimes the ρ is omitted from notation. A finite-dimensional representation V together with a basis of V is a matrix representation (see problem 7 on HW #1).

Let (ρ, V) and (π, W) be two representations of G over K. A *homomorphism* from (ρ, V) to (σ, W) is a K-linear transformation $f : V \to W$ that is G-*equivariant*: that is, for each $g \in G$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

A homomorphism of representations of G (or G-*representations*) is an *isomorphism* if it is an isomorphism of underlying vector spaces.

If (ρ, V) is a representation of G over K, then a K-linear subspace W of V is a *subrepresentation* if it is stable by the action of G. (Such a *G-stable* W is also sometimes called *G-invariant*, but note that the action of G on W need not be trivial.) The representation (ρ, V) is *irreducible* if it has no proper subrepresentations. It is *decomposable* if $V = W \oplus U$, where $W, U \subset V$ are proper subrepresentations; otherwise it is *indecomposable*. It is *totally decomposable* or *completely reducible* if V is an (internal) direct sum of irreducible subrepresentations: $V = \bigoplus_i W_i$, where $W_i \subseteq V$ are irreducible.

Below we take $K = \mathbb{C}$.

(a) If $f : V \to W$ is a homomorphism of G-representations, then $\ker f$ is a subrepresentation of V and $\operatorname{im} f$ is a subrepresentation of W.

(b) If W is a subrepresentation of a representation (ρ, V) of a group G, then the quotient space V/W carries a representation of G inherited from V, and the projection $V \to V/W$ is a homomorphism of G-representations.

(c) Show that the map $\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ given by $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ defines a reducible but not decomposable representation. What can you say more generally about the family of maps $1 \mapsto \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix}$ for any $\alpha \in \mathbb{C}^\times$?

(d) Let (ρ, V) be a finite-dimensional representation of a finite group G, and let $W \subset V$ be a subrepresentation. Show W has a *complement* in V: that is, show that there is another subrepresentation $U \subset V$ with $V = W \oplus U$.

To do this, let U_0 be any vector-space complement to W (that is, U_0 need not be G-stable) and let $\pi_0 : V \to W$ be the projection of V onto W with kernel U_0. Said another way, π_0 be any left inverse (*retraction*) of the inclusion $\iota : W \hookrightarrow V$ as complex vector spaces, corresponding to any splitting of the exact sequence

(1)
$$0 \to W \xrightarrow{\iota} V \longrightarrow V/W \longrightarrow 0.$$

of $\mathbb{C}$-vector spaces. (Problem 3 on HW #3 may be helpful.)

Show that

$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi_0 \circ \rho(g^{-1})$$

is a G-equivariant left inverse of ι: that is, a splitting of (1) as G-representations. Why does this mean that W has a complement in V?

(e) Show that any complex finite-dimensional representation of a finite group is completely reducible. Would your argument still work for $K = \mathbb{R}$? For $K = \mathbb{Q}$? For $K = \mathbb{F}_p$?

(5) **Group algebras:** Let A be a commutative ring and G a group. We construct a new ring $A[G]$. As a A-module, this ring is free with basis $\{g : g \in G\}$ indexed by the elements of G: that is, elements of $A[G]$ are formal linear combinations $\sum_{g\in G} a_g g$ with only finitely many of the coefficients a_g nonzero. Define multiplication via $g \cdot h = gh$ and extend A-bilinearly. Convince yourself that $A[G]$ is an A-algebra, called the *group algebra* of G over A. The elements g of $A[G]$ are the *group-like elements*. What is the multiplicative identity of $A[G]$? Is $A[G]$ commutative?

(a) Show that $A[\mathbb{Z}/n\mathbb{Z}] \cong A[x]/(x^n - 1)$. Show that $A[\mathbb{Z}] \cong A[x, x^{-1}]$, the ring of *Laurent polynomials* over A.

(b) Is $\mathbb{R}[Q_8]$ isomorphic to the division algebra of real Hamiltonians $\mathbb{H}$? Prove or explain.

(c) Show that association $G \rightsquigarrow \mathbb{Z}[G]$ is a functor **Group** $\to$ **Ring**, and is left adjoint to the group-of-units functor taking a ring R to $R^\times$.

(d) Let K be a field. If (ρ, V) is a representation of G over K, show that V has the structure of a $K[G]$-module via $g \cdot v := \rho(g)v$. Conversely, show that any $K[G]$-module is a representation of G over K. Show that there is an equivalence of categories between the category of $K[G]$-modules and the category of representations of G over K.

Considering $K[G]$ as a (left) module over itself, we see that it carries a permutation representation of G: this is the *(left) regular* representation.

(e) Now let $K = \mathbb{C}$, and assume that G is a finite group. Let $\chi : G \to \mathbb{C}^\times$ be a character. Find a complex line in in the $\mathbb{C}$-vector space $\mathbb{C}[G]$ on which G acts by χ. How many different such lines are there?