MA 741: Algebra I / Fall 2020
Homework assignment #8
Due week of December 1, 2020, or before the end of the semester

Typos in (3) corrected 12/11/2020.

To turn in your work, please email your well-titled document (title should identify you, this course, and the HW set number) to `buma741fall2020@gmail.com` with "HW 8" in the subject line. Please indicate with whom you worked on the problem set.

(1) **Localization of modules:** Let A be a commutative ring, and M an A-module. Let $S \subset A$ be a multiplicatively closed subset (see 3 on HW #6). As before, the relation $M \times S$ given by

$$(m, s) \equiv (m', s') \text{ if there exists } t \in S \text{ such that } t(ms' - sm') = 0$$

is an equivalence relation (check this); we write $\frac{m}{s}$ for the equivalence class of (m, s) and let $S^{-1}M$ be the set of equivalence classes. Convince yourself that $S^{-1}M$ is an $S^{-1}A$-module: that is, addition in $S^{-1}M$ and scalar multiplication by $S^{-1}A$ are well defined and behave as expected.

(a) Show that the association $M \rightsquigarrow S^{-1}M$ is an exact functor A-**mod** $\to (S^{-1}A)$-**mod**.

(b) Show that map

$$\begin{aligned} S^{-1}A \otimes_A M &\to S^{-1}M \\ \tfrac{a}{s} \otimes m &\mapsto \tfrac{am}{s} \end{aligned}$$

is an isomorphism of $(S^{-1}A)$-modules. Conclude that $S^{-1}A$ is a flat A-module.

(*Hint:* For injectivity, it may be helpful to start by showing that every element of $S^{-1}A \otimes_A M$ is represented by a pure tensor: that is, has the form $\frac{a}{s} \otimes m$.)

If $\mathfrak{p}$ is a prime ideal of A and $S = A - \mathfrak{p}$, write $M_\mathfrak{p}$ for $S^{-1}M$, the *localization* of M at $\mathfrak{p}$. A property of an A-module (respectively, of an A-module homomorphism) is called a *local property* if the property holds for an A-module M (respectively, of a A-module homomorphism $\varphi : M \to N$) if and only if the property holds for the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ (respectively, for the $A_\mathfrak{p}$ module homomorphism $\varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$) for every prime ideal $\mathfrak{p}$ of A.

(c) Show that being zero is a local property of A-modules: that is, show that $M = 0$ and only if $M_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ of A.

(*Hint:* For any $m \in M$, the *annihilator* of m is the subset $\mathrm{Ann}(m) = \{a \in A : am = 0\}$; this is an ideal of A (why?). When is $\mathrm{Ann}(m)$ proper?)

(d) Show that being injective is a local property of A-module homomorphisms. (That is, show that $\varphi : M \to N$ is injective if and only if $\varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for every prime ideal $\mathfrak{p}$ of A.)

(e) Show that being surjective is a local property of A-module homomorphisms.

(f) **Optional:** Show that being flat is a local property of A-modules.

(2) **Snake lemma:** Let R be a ring and suppose that we have a homomorphism of two short exact sequences of R-modules:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[r, "g"] \arrow[d, "\beta"] & P \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r] & 0. \end{tikzcd}$$

That is, $0 \to M \to N \to P \to 0$ and $0 \to M' \to N' \to P' \to 0$ are short exact, and α, β, γ are maps between corresponding R-modules so that the diagram above commutes.

Show that this setup induces a naturally defined long exact sequence

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0,$$

where the *connecting homomorphism* δ sends an element $p \in \ker\gamma \subseteq P$ to the $(\operatorname{im}\alpha)$-coset of the preimage in M' of $\beta(n)$ for any preimage n of p in N; you'll have to explain why this makes sense and is well defined.

(3) **Inverse limits:** Let I be a poset. Suppose that for every $i \in I$ we have an object G_i in a category $\mathbf{C}$, and for every pair of indices $i \leq j$ we have a morphism $\mu_{ji} : G_j \to G_i$ with the properties that (a) $\mu_{ii} = \mathrm{id}_{G_i}$ for all $i \in I$ and (b) $\mu_{ji} \circ \mu_{kj} = \mu_{ki}$ whenever $i \leq j \leq k$. In other words we have the data of a contravariant functor from I thought of as a category to $\mathbf{C}$.

The *inverse limit* of this *inverse system* (more precisely, of this functor $I^{\mathrm{op}} \to \mathbf{C}$), denoted $\varprojlim G_i$, is an object P in $\mathbf{C}$ equipped with morphisms $\pi_i : P \to G_i$ for each $i \in I$ and satisfying the following (final) universal mapping property: if X any object in $\mathbf{C}$ with morphisms $f_i : X \to G_i$ with $f_i = \mu_{ji} \circ f_j$ whenever $i \leq j$, then there is a unique morphism $\beta : X \to P$ factoring each f_i: that is, $\pi_i \circ \beta = f_i$ for each $i \in I$. As usual, if P exists then it is unique up to unique isomorphism.

(a) Let $\mathbf{C}$ be the category of groups and $\{G_i, \mu_{ji} : G_j \to G_i\}$ an inverse system of groups as above. Consider the following subset of the direct product $\prod_{i \in I} G_i$:

$$P := \{(g_i)_{i \in I} : \mu_{ji}(g_j) = g_i \text{ for all } i \leq j \text{ in } I\}.$$

Show that P is a group satisfying the inverse limit property above: that is, $P = \varprojlim G_i$. Show that the same construction gives the inverse limit in the categories of abelian groups, commutative rings, and R-modules for a ring R.

Now let p be a prime, and consider the following inverse system of commutative rings on the poset $\mathbb{Z}^+$ with its usual $\leq$ relation. For every $n \in \mathbb{Z}^+$, let $G_n = \mathbb{Z}/p^n\mathbb{Z}$, and for $n \leq m$ let $\mu_{mn} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be the reduction-modulo-p^n map. The inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is called the ring of *p-adic integers*, denoted by $\mathbb{Z}_p$.

(b) Show that every element a of $\mathbb{Z}_p$ has a unique infinite p-adic expansion

$$a = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots, .\text{where } 0 \leq a_n < p.$$

In particular, $\mathbb{Z}_p$ is uncountable. Describe how these expansions may be used to add and multiply elements of $\mathbb{Z}_p$.

(c) In the notation of (3b), show that $a \in \mathbb{Z}_p$ is unit if and only if $a_0 \neq 0$. Show that $\mathbb{Z}_p$ is an integral domain containing a copy of $\mathbb{Z}$ and of $\mathbb{Z}_{(p)}$. (See 5(d) on HW #6 for the definition of $\mathbb{Z}_{(p)}$.) Show that the ideal $p\mathbb{Z}_p$ is maximal, and is the only maximal ideal of $p\mathbb{Z}_p$.

(d) What is the p-adic expansion of -1? Of $\frac{1}{1-p}$? Of $\frac{1}{2}$ if p is odd?

(4) Let K be a field, and consider the formal derivative operation on $K[x]$: namely, given $f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in K[x]$, set $f' := a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$.

(a) Show that $f \mapsto f'$ is a *derivation*: it's a K-linear map $K[x] \to K[x]$ satisfying the Leibniz rule: $(fg)' = f'g + fg'$.

(b) Let $f \in K[x]$, and suppose L/K is an extension where f splits completely: that is, $f = (x - \alpha_1) \cdots (x - \alpha_n) \in L[x]$ for some $\alpha_1, \cdots, \alpha_n$ in L. (For example, L could be an algebraic closure $\overline{K}$ of K.) Show that f and f' are coprime if and only if $\alpha_1, \cdots, \alpha_n$ are distinct.

(Recall that two polynomials f and g in $K[x]$ are coprime if (f, g) is the unit ideal. Equivalently, f and g are coprime if their gcd, which can be computed using Euclid's algorithm, is is in $K^\times$. In particular, being coprime in $K[x]$ is the same as being coprime in $L[x]$ for any extension L/K.)

Now let p be a prime, and let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. For any $n \geq 1$ consider the subset

$$F_n := \{\alpha \in \overline{\mathbb{F}}_p : \alpha^{p^n} = \alpha\} \subset \overline{\mathbb{F}}_p.$$

(c) Show that F_n has exactly p^n elements. (*Hint:* F_n is the set of roots of $x^{p^n} - x$.) Show that F_n forms a subfield of $\overline{\mathbb{F}}_p$ containing $\mathbb{F}_p$.

(d) Prove that $F_m \subseteq F_n$ if and only if $m \mid n$.

(*Hint:* It may be helpful to show that $(p^m - 1) \mid (p^n - 1)$ if $m \mid n$.)

(e) Explain why $\mathbb{F}_p[x]$ has at least one irreducible polynomial of degree n for every $n \geq 1$. Must there be more than one for every p, n?

(5) **Induction of a character from an index-2 subgroup:** Let G be a group, and $H \subset G$ an index-2 subgroup. Let $\psi : H \to \mathbb{C}^\times$ be a character, and fix an element $c \in G - H$. Consider the following map $\rho : G \to \mathrm{GL}_2(\mathbb{C})$:

$$\rho(g) := \begin{cases} \begin{pmatrix} \psi(g) & 0 \\ 0 & {}^c\psi(g) \end{pmatrix} & \text{if } g \in H; \\ \\ \begin{pmatrix} 0 & \psi(gc) \\ \psi(c^{-1}g) & 0 \end{pmatrix} & \text{otherwise.} \end{cases}.$$

(See problem 6 on HW #6 for the definition of ${}^c\psi$.)

(a) Show that this map defines a representation of G on $V = \mathbb{C}^2$. This is the *induced representation* $V = \mathrm{Ind}_G^H \psi$.

(b) Show that $\mathrm{Ind}_H^G \psi$ is irreducible if and only if $\psi \neq {}^c\psi$.

(c) Construct an explicit isomorphism between $\mathrm{Ind}_H^G \psi$ and $\mathrm{Ind}_H^G {}^c\psi$.

(d) Let $A_3 \subset S_3$ be the index-2 subgroup, and consider the character $\chi : A_3 \to \mathbb{C}^\times$ sending a generator (to fix ideas, say, the cycle $(1\ 2\ 3)$) to $\omega = e^{\frac{2\pi i}{3}}$. Let $(\rho, V) = \mathrm{Ind}_{A_3}^{S_3} \chi$. Compare this representation to the two 2-dimensional real representations of S_3 that you constructed in (7a) and (7e) of HW #1.

(Can you show that all three are isomorphic as representations of S_3? Here of course we're assuming that we've extended scalars from $\mathbb{R}$ to $\mathbb{C}$ on the representations from HW #1.)

(e) **Optional challenge problem:** Can you find a copy of $\mathrm{Ind}_{A_3}^{S_3} \chi$ as a subrepresentation of the left regular representation $\mathbb{C}[S_3]$? (And if yes, how many copies of $\mathrm{Ind}_{A_3}^{S_3} \chi$ can you find inside $\mathbb{C}[S_3]$?)

Additional optional problem

(6) Recall that an **Ab**-category is one where Hom-sets are abelian groups and composition distributes over the group operation (such a category is also called *preadditive* in the literature). A functor between **Ab**-categories is *additive* if the maps on Hom-sets are morphisms of abelian groups.

(a) If $\mathbf{C}$ and $\mathbf{D}$ are **Ab**-categories and $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is an additive functor, show that $\mathcal{F} : \mathrm{End}_{\mathbf{C}}(A) \to \mathrm{End}_{\mathbf{D}}(\mathcal{F}(A))$ is a ring homomorphism for any object A of $\mathbf{C}$.

Recall also that in a category with a zero object, every Hom-set has a distinguished zero morphism passing through the zero object. Let $\mathbf{C}$ be a category with a zero object; for any A, B in $\mathbf{C}$, write 0_{AB} for the corresponding zero morphism in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

(b) If A, B, C, D are in $\mathbf{C}$, show that $0_{BC} \circ f = 0_{AC}$ and $g \circ 0_{BC} = 0_{BD}$ for any arrows $f : A \to B$ and $g : C \to D$ in $\mathbf{C}$.

Now let $\mathbf{C}$ and $\mathbf{D}$ be two $\mathbf{Ab}$-categories with zero objects.

(c) Show that 0_{AB} is the additive identity of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ for any A, B in $\mathbf{C}$.

(d) If 0 is a zero object of $\mathbf{C}$ and $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is an additive functor, show that $\mathcal{F}(0)$ is a zero object of $\mathbf{D}$. (*Hint:* Use (6a).)

An $\mathbf{Ab}$-category with zero objects and finite products is called an *additive* category.

(e) Show that in an additive category, finite products are also coproducts.

(f) Show that an additive functor between two additive categories sends finite coproducts to finite coproducts. (Note that the zero object is the coproduct of the empty collection of objects, so that (6d) can be seen as a special case.)