8. DESCENT AND CANONICAL MODELS.

- **RECALL:** Shimura varieties:
  - $G$ - reductive group over $\mathbb{Q}$.
  - $X$ - conjugacy classes of $h: S \to G_{\mathbb{R}}$.
  - $K$ - compact open $\subset G(\mathbb{A}_{\mathbb{F}})$.

\[
Sh_K(G, X) = \frac{G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\mathbb{F}})}{K} \quad \text{(a variety)}
\]

\[
Sh(G, X) = \lim_{\rightarrow K} Sh_K(G, X) \quad \text{(a scheme/pro-variety)}
\]

1. **Galois Descent**

- **Q:** If $X/\mathbb{C}$ is a variety, is there some $k \subset \mathbb{C}$, $X_k/k$ s.t. $X \cong X_k \times \mathbb{C}$?

In that case, we say that $X$ descends to $k$ or $X_k$ is a model for $X/k$.

**EXAMPLE**

- $C: x^2 + y^2 = \pi / \mathbb{C}$.

Is there a model over $\mathbb{Q}$?

- **Yes:** $C \cong C^1: (x')^2 + (y')^2 = 1 / \mathbb{C}$ \quad \(x' = \frac{x}{\sqrt{\pi}}, \quad y' = \frac{y}{\sqrt{\pi}}\)

$C^1 = C_0 \times \mathbb{C}$, where $C_0: x^2 + y^2 = 1 / \mathbb{Q}$. 

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$E: y^2 = x^3 + ix + 1.$

Is there a model over $\mathbb{Q}$?

No: $j(E) = 1728 \frac{4i^3}{4i^3 + 27} \neq \mathbb{Q},$

but if $E_0/\mathbb{Q}$ then $j(E_0) \in \mathbb{Q}.$

$j(E_0 \times \mathbb{C}).$

Note: for $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}),$

$E^\sigma: y^2 = x^3 + \sigma(i)x + 1$

If we had a model $E_0/\mathbb{Q}$, then

$E^\sigma \cong E_0 \times \mathbb{C} \cong E \Rightarrow E^\sigma \cong E$

But $j(E^\sigma) = j(E)^\sigma \neq j(E)$ if, say, $\sigma$ is complex conjugation.

In general, we have this necessary condition:

- **Condition 1:** If a model of $C/k$ exists, we should have

$$C \xrightarrow{f_\sigma} C^\sigma \quad \forall \sigma \in \text{Gal}(C/k).$$

- **Remark:** Models over $k$ are not unique (example: $E \cong E_4$)

- **Remark:** The necessary condition 1 is sufficient in genus 1.

Check: If $f_\sigma: E \xrightarrow{\sim} E^\sigma \quad \forall \sigma \in \text{Gal}(C/k)$, then

$\sigma(j(E)) = j(E^\sigma) = j(E) \quad \forall \sigma$

$\Rightarrow j(E) \in k.$ So $E_0/k$ with $j$-invariant $j(E).$
Now we switch to quasi-projective varieties \( X / \mathbb{C} \).

If there exists a model \( X_0 / \mathbb{k} \), then we have various natural isomorphic curves:

\[
X = X_0 \times \mathbb{C} \xrightarrow{\sim} (X_0 \times \mathbb{C})^\sigma = X, \forall \sigma \in \text{Gal}(\mathbb{C}/\mathbb{k}),
\]

and we have the relations:

\[
f_\sigma \circ f_\tau = f_{\sigma \tau} \quad \forall \sigma, \tau \in \text{Gal}(\mathbb{C}/\mathbb{k}).
\]

**Proposition (Weil 1956)**

\( X / \mathbb{C} \) descends to \( \mathbb{k} \) (\( \Rightarrow \))

\( \Leftrightarrow \exists f_\sigma : X \xrightarrow{\sim} X^\sigma \forall \sigma \in \text{Gal}(\mathbb{C}/\mathbb{k}) \) satisfying the cocycle condition:

\[
f_{\sigma \tau} = f_\sigma \circ f_\tau : X \to X^\sigma \to X^{\sigma \tau}.
\]

Such a system is called a Weil descent datum.

**Remark:** If \( X \) has only the trivial automorphism, then there is no way for the cocycle condition to fail. In this case, the necessary condition \( 1 \) is sufficient.

**Definition.**

The field of moduli of \( X / \mathbb{C} \) is the fixed field of

\[
\{ \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \mid \exists X \xrightarrow{\sim} X^\sigma \}.
\]

This is the smallest field where we may be able to descend \( X \) to.
Let $m$ be odd and define a hyperelliptic curve of genus $m-1$ as

$$y^2 = a_0 x^m + \sum_{r=1}^{m} (a_r x^{m+r} + (-1)^r a_r^s x^{m-r}), \quad a_i \in \mathbb{C}, a_0 \in \mathbb{R}, \quad a_m = 1,$$

where $^s$ is complex conjugation.

As long as all $a_i, a_i^s$ are algebraically independent over $\mathbb{Q}$, there are no automorphisms except $\text{id}$.

Note that there is an isomorphism

$$f_g : X \rightarrow X^s$$

$$(x, y) \mapsto (-x^{-1}, i x^{-m} y),$$

so the field of moduli is contained in $\mathbb{R}$.

However, $f_g \circ f_g$ is the hyperelliptic involution (i.e., $f_g \circ f_g (x, y) = (x, -y)$), so the cocycle condition fails ($f_g \circ f_g \neq f_g^2 = f_{\text{id}} = \text{id}$).

Therefore, $X$ does not descend to $\mathbb{R}$.

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**Theorem (Weil 1956. Reference: Milne, 14.6).**

$X/\mathbb{C}$ descends to $k$ if all $X^s \cong X$ and there exists a set of points $P_1, \ldots, P_n \in X(\mathbb{C})$ such that:

1) Any automorphism of $X$ fixing all the $P_i$'s is the identity.

2) There is a subfield $L \subseteq \mathbb{C}$, finitely generated over $k$, such that

$$\sigma P_i = P_i \quad \forall \sigma \in \text{Gal}(\mathbb{C}/L).$$

**Goal:** Identify a special set of points on which we know the Galois action, as well as some field $L$ as above.
2. Canonical models.

2.1. Special points.

Imaginary quadratic integers in $\mathbb{H}^+$ are special, in the sense that they are fixed by some elliptic matrix $y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{R}) \cap \mathbb{H}^+$. (Recall that $y$ is elliptic if $(tr y)^2 - 4 \det y < 0$).

**Definition.**

An algebraic torus over $k$ is an algebraic group $T$ s.t. $T_{\bar{k}} \cong \mathbb{G}_m^n$.

**Definition.**

Let $(G, X)$ be a Shimura datum. A special point is some $x \in X$ s.t. $\exists$ $\mathbb{Q}$-torus $T \subseteq G$ such that $h_x(\mathbb{C}^*) \leq T(\mathbb{R})$.

We also say that $(T, x)$ is a special pair.

**Remark:** $(T, x)$ special means that $T(\mathbb{R})$ acting by conjugation fixes $x$. Conversely, if $T \subseteq G$ is a maximal torus and $T(\mathbb{R})$ fixes $x$, then $(T, x)$ is special.

**Example.**

Let $G = \text{GL}_2$ and $\mathbb{H}^+ = \mathbb{C} \setminus \mathbb{R}$. Then $G(\mathbb{R}) \cap \mathbb{H}^+$ acts via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}.$$

If $z \in \mathbb{H}^+$ generates an imaginary quadratic field $E / \mathbb{Q}$, then we can embed $E \to \text{Mat}_2(\mathbb{Q})$ using the basis $\{1, -z\bar{z}\}$ for $E$.

We get a maximal torus $T := \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \subseteq G$.

**Exercise:** find $x$ such that $(T, x)$ is a special pair.
2.2. Canonical models.

Given a special pair \((T, x) \in (G, X)\), we have a cocharacter \(\mu_x^E\) defined over \(E(x)\), and we can form the map:

\[
\Gamma_x : \mathbb{A}^E_{x(x)} \xrightarrow{\pi} \mathbb{A}^E_{x(x)} \xrightarrow{\text{proj}} T(A_{E}) \xrightarrow{\text{projection}} T(A_{E}).
\]

We have the Artin map from Class Field Theory:

\[
\text{art}_{E(x)} : \mathbb{A}^E_{x(x)} \rightarrow \text{Gal}(E(x)^{ab}/E(x))
\]

Call \([x, a]_K\) the point of \(\text{Sh}_K(G, X)\) represented by \((x, a), a \in G(A_{E})\).

**Definition.** (Milne 12.8)

Let \((G, X)\) be a Shimura datum and \(K \subset G(A_{E})\) a compact open subgroup. A model \(M_K(G, X)\) of \(\text{Sh}_K(G, X)\) over the reflex field \(E(G, X)\) is a canonical model if for all special pair \((T, x) \in (G, X)\) and \(a \in G(A_{E})\), \([x, a]_K\) has coordinates in \(E(x)^{ab}\) and

\[
\sigma [x, a]_K = [x, \sigma(x) a]_K
\]

for all \(\sigma \in \text{Gal}(E(x)^{ab}/E(x))\) and \(s \in \mathbb{A}^E_{x(x)}\) s.t. \(\text{art}_{E(x)}(s) = \sigma\).

**Langlands Conjecture.** (Milne 1983)

Let \((G, X)\) be a Shimura datum and \(\sigma \in \text{Aut}(\mathbb{C})\). Langlands defined \((G^\sigma, X^\sigma)\) and conjectured a unique isomorphism

\[
f_\sigma : \text{Sh}(G^\sigma, X^\sigma) \rightarrow \text{Sh}(G, X)
\]

satisfying some conditions.

**Theorem.** (Milne 1983)

For any Shimura datum \((G, X)\), \(\text{Sh}(G, X)\) has a canonical model (defined to be a compatible system of canonical models for \(\text{Sh}_K(G, X)\)). This model is unique up to unique isomorphism.