**Universal Deformation: Properties.**

- We've been trying to understand deformations of $\overline{\pi}: \Pi \to \text{GL}_n(k)$, $k$ finite field of char. $p$, $\Pi$ profinite group satisfying $\Phi_p$.
  
  (E.g., $\Pi = \text{G}_{K^s}$ or $\text{G}_{K,s}$).

- $\mathcal{C} = \text{category of complete local Noetherian rings with residue field } k$.
  
  $\mathcal{C}^\circ = \text{subcategory of } \mathcal{C} \text{ consisting of Artinian rings}$.

  Similarly, for each $\Lambda \in \mathcal{C}$, we defined the analogous categories $\mathcal{C}_{\Lambda}, \mathcal{C}_{\Lambda}^\circ$ of $\Lambda$-algebras.

- We have $D_{\overline{\pi}}: \mathcal{C}_{\Lambda}^\circ \to \text{Sets}$, the deformation functor associated to $\overline{\pi}$, whose value at $R$ is the set of $R$-deformations of $\overline{\pi}$, i.e., $\Psi: \Pi \to \text{GL}_n(R)$.

- **Last time:** $D_{\overline{\pi}}$ is pro-representable by $R = R(\Pi, \overline{\pi}, k) \in \mathcal{C}_{\Lambda}$, with universal deformation $\mathcal{P}: \Pi \to \text{GL}_n(R)$ such that all deformations of $\overline{\pi}$ to $\Lambda \in \mathcal{C}_{\Lambda}^\circ$ arise via unique $R \to \Lambda$

  (assuming $C(\overline{\pi}) = \text{Hom}_\Pi(k^n, k^n) = \{ P \in M_n(k) \mid \overline{\pi}(g)P \overline{\pi}(g) = P \text{ for all } g \}$ is $k$).

- The existence of $R$ implies some functorial properties of $D_{\overline{\pi}}$, $\Lambda = \mathcal{D}_{\Lambda}$

  **Example** $\text{det}: \text{GL}_n \to \text{GL}_1$ sends deformations of $\overline{\pi}$ to deformations of $\text{det}(\overline{\pi})$, so we get a homomorphism from the completed group ring $\Lambda[[\Gamma]] \to R(\overline{\pi})$,

  where $\Gamma = \Pi^{\text{ab}, (p)}$ since $\Lambda[[\Gamma]] = R(\text{det} \overline{\pi})$. 

EXAMPLE: If $\bar{g}$ is equivalent to $\bar{g}'$ by $X \in \text{GL}_n(k)$, then we get a map $R(\bar{g}) \to R(\bar{g}')$, which is an isomorphism and doesn't depend on the choice of $X$.

**Tangent Spaces and Cohomology Groups.**

Fix $\bar{g}$, $\Lambda$. Then the tangent space to $D := D_{\bar{g}}, \Lambda$ is

$$t_D := D(k[\varepsilon]).$$

Since $D$ is pro-representable, we see that

$$t_D = \text{Hom}_\Lambda (R, k[\varepsilon]) = \text{Hom}_k (\Lambda R/(\Lambda R, \Lambda \Lambda), k).$$

Working with our deformation functor $D$, we can go further:

Suppose $\bar{g}(g) = a \in \text{GL}_n(k)$. Then if $\bar{g}_1$ is a deformation of $\bar{g}$ to $k[\varepsilon]$ then

$$\bar{g}_1(g) = (1 + bg \varepsilon)a$$

for some $bg \in H_n(k)$.

**Claim:** $bg$ is a 1-cocycle for the $\text{Ad}$-representation $\text{Ad}(\bar{g})$,

where $\text{Ad}(\bar{g}) = H_n(k)$ with action $g \cdot s = \bar{g}(g) \cdot s \cdot \bar{g}(g)^{-1}$.

The condition that $\bar{g}_1$ is a group homomorphism means

$$\bar{g}_1(gh) = (1 + bg \varepsilon)x \cdot y,$$

where $\bar{g}(g) = x$, $\bar{g}(h) = y$.

Thus

$$\bar{g}_1(g) \bar{g}_1(h) = (1 + bg \varepsilon)x(1 + bh \varepsilon)y = (x + bg \varepsilon x)(1 + bh \varepsilon y) =$$

$$(x + bg \varepsilon x + bh \varepsilon y) = (1 + bg \varepsilon + bh \varepsilon x' \varepsilon)y \Rightarrow bg + bh \varepsilon x' = bg + gbh \varepsilon x'$$
This gives a correspondence between
defining \( \mathcal{G} \) to \( k[x] \leftrightarrow \) 1-cocycles on \( \text{Ad}(\mathcal{G}) \).

So \( t_0 \to H'(\mathbb{T}, \text{Ad}(\mathcal{G})) \) is an isomorphism of \( k \)-vector spaces.

(Exercise: check these claims).

**Corollary**

If \( d_1 = \dim_k H'(\mathbb{T}, \text{Ad}(\mathcal{G})) \) then \( R \) is a quotient of a power series ring in \( d_1 \) variables, i.e.,

\[
0 \to I \to \bigwedge[\ldots, X_d] \to R \to 0
\]

We can also describe \( t_0 \) with data intrinsic to \( \mathcal{G} \) as follows:

Let \( V_{\mathcal{G}} \) be the space corresponding to \( \mathcal{G} \). Let

\[
0 \to V_{\mathcal{G}} \to E \to V_{\mathcal{G}} \to 0
\]

be an extension of \( V_{\mathcal{G}} \) by \( V_{\mathcal{G}} \) in the category of \( k[\mathbb{T}] \)-modules.

Then \( s_E : \mathbb{T} \to GL_n(k) \) (the representation corresponding to \( E \))

can be expressed as

\[
s_E(g) = \begin{bmatrix}
  \mathcal{G}(g) & A_g \\
  0 & \mathcal{G}(g)
\end{bmatrix}
\]

for some \( A_g \in H^0(k) \).

One can show that \( g \to A_g \mathcal{G}(g)^{-1} \) is a 1-cocycle on \( \text{Ad}(\mathcal{G}) \).

Thus it induces an isomorphism

\[
\text{Ext}_k^1(V_{\mathcal{G}}, V_{\mathcal{G}}) \cong H'(\mathbb{T}, \text{Ad}(\mathcal{G}))
\]
Obstructed and Unobstructed Deformation Problems

Suppose $R_i, R_0 \in \mathbb{C}^n$ with a surjective map $R_i \to R_0$ with kernel $I$ satisfying $I \cdot M_{R_i} = 0$ (i.e., $I$ is an $R_i/IM_{R_i}$ vector space).

Given $\varphi : \Pi \to \text{GL}_n(R_0)$, we want to measure obstructions to lifting to a map $\gamma : \Pi \to \text{GL}_n(R_i)$ (i.e., $\gamma \mod I = \varphi$).

For $\gamma$ to be a homomorphism, we need

$$c(g_1, g_2) = \gamma(g_1) \gamma(g_2) \gamma(g_1)^{-1} \gamma(g_2)^{-1}$$

to equal $1$ always.

But we know that

$$c(g_1, g_2) = 1 + d(g_1, g_2) \quad \text{for} \quad d(g_1, g_2) \in H^0(I) = H_0(\mathbb{C}) \otimes I.$$ 

**Claim:** $d(g_1, g_2)$ is a $2$-cocycle. [And replacing $\gamma$ by a different $\tilde{\varphi}$]

This gives $0(\varphi) \in H^2(\Pi, \text{Ad}(\varphi) \otimes I) \cong H^2(\Pi, \text{Ad}(\bar{\varphi})) \otimes I$.

We expect the deformation problem to be simpler when $H^2(\Pi, \text{Ad}(\varphi)) = 0$.

**Theorem (Haraux)**

Assume $\mathcal{C}(\varphi) = k$. Let $R := R(\Pi, \bar{\varphi}, k).

Set $d_1 := \dim_k H^0(\Pi, \text{Ad}(\bar{\varphi}))$. Then

$$\dim \left( R/IM_{\Lambda} R \right) \geq d_1 - d_2.$$ 

(Where $\dim = \text{Krull dimension}$).

Furthermore, if $d_2 = 0$, then $\dim \left(R/IM_{\Lambda} R \right) = d_1$, and in fact

$$R \cong \Lambda \left[ X_1, \ldots, X_{d_1} \right].$$
**DIMENSION CONJECTURE**

When $\bar{f}$ is absolutely irreducible, we have

$$\dim \left( \mathcal{R}/m_{\bar{f}} \mathcal{R} \right) = d_1 - d_2.$$

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**Galois Representations.**

Let $K$ be a number field, $d := [K: \mathbb{Q}]$,

- $S =$ finite set of places in $K$,
- $\Sigma =$ set of infinite places.

Set $T = G_{K,s}$. Let $\bar{\rho} : G_{K,s} \to \text{GL}_n(k)$ be such that $\text{L}(\bar{\rho}) = k$ with deformation ring $\mathcal{R}$.

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**Tate’s Global Euler Characteristic Formula.**

Let $M$ be a finite $G_{K,s}$-module. Then, if $1$.

$$\frac{\# H^i(G_{K,s}, H)}{\# H^i(G_{K,s}, M)} = \frac{1}{\prod_{\text{rectifiable primes } \mathfrak{p} \in S_{\infty}} \# H^0(G_{K_{\mathfrak{p}}}, H)}.$$

Taking $M = \text{Ad}(\bar{\rho})$, of size a power of $p$, and requiring $S$ to contain all places with $p$, Tate’s formula translates to:

$$h^0(G_{K,s}, \text{Ad}(\bar{\rho})) - h^i(G_{K,s}, \text{Ad}(\bar{\rho})) + h^2(G_{K,s}, \text{Ad}(\bar{\rho})) =$$

$$= \sum_{\mathfrak{p} \in S_{\infty}} h^0(G_{K_{\mathfrak{p}}}, \text{Ad}(\bar{\rho})) - d \cdot \dim_k \text{Ad}(\bar{\rho}).$$
I.e.,
\[ d_1 - d_2 = d_0 + d \cdot n^2 - \sum_{\omega \in \mathfrak{S}_n} h^\omega(G_{k,\mathfrak{s}}, \text{Ad}(\bar{\varphi})). \]

But \( d_0 = \dim H^0(G_{k,\mathfrak{s}}, \text{Ad}(\bar{\varphi})) = \dim (\text{Ad}(\bar{\varphi}) G_{k,\mathfrak{s}} = 1, \text{ since} \)
\( \text{Ad}(\bar{\varphi}) G_{k,\mathfrak{s}} = \mathcal{Z}(\bar{\varphi}) = k. \)

We get:

**PROPOSITION.**

Let \( K/\mathbb{Q} \) be a number field of degree \( d, \)

\( \bar{\varphi}: G_{k,\mathfrak{s}} \to \text{GL}_n(k), \text{ s.t. } \mathcal{Z}(\bar{\varphi}) = k; \text{ R its universal deformation ring} \)

Then
\[ \dim (R/m_{\mathfrak{m}} R) \geq 1 + d \cdot n^2 - \sum_{\omega \in \mathfrak{S}_n} h^\omega(G_{k,\mathfrak{s}}, \text{Ad}(\bar{\varphi})). \]

**EXAMPLE**

If \( \bar{\varphi} \) is a character, then \( R = \Lambda \left[ G_{k,\mathfrak{s}}^{ab,r} \right], \text{ and} \)

\[ \frac{R}{m_{\mathfrak{m}} R} = k \left[ G_{k,\mathfrak{s}}^{ab,r} \right]. \]

Hence its dimension as a ring is

the rank \( r \) of \( G_{k,\mathfrak{s}}^{ab,r} \) as a \( \mathbb{Z}_p \)-module, i.e., it is

\[ \mathbf{rk}_{\mathbb{Z}_p} \text{Hom}_{\mathcal{Cts}}(G_{k,\mathfrak{s}}, \mathbb{Z}_p). \]

The RHS of the proposition inequality is \( 1 + r \).

Each \( h^\omega(G_{k,\mathfrak{s}}, \text{Ad}(\bar{\varphi})) \leq 1 \)

\[ \Rightarrow d - r, -r, = r. \]

Lazard's conjecture: this is an equality.