## Vector space properties

# MA 751
# Part 1

## Linear Algebra, functional analysis

This material is from basic linear algebra as a reference - will not go over in detail in class.

## 1. Preliminaries:

**Recall:** $\mathbb{R}$ = {real numbers}; $\mathbb{R}^3$ = {all triples of real numbers}, etc.

# Examples

**Definition (part 1)**: A *vector space* is a set of objects $V$ on which addition and scalar multiplication are defined.

Precisely if $\mathbf{v}_1, \mathbf{v}_2 \in V$ we define

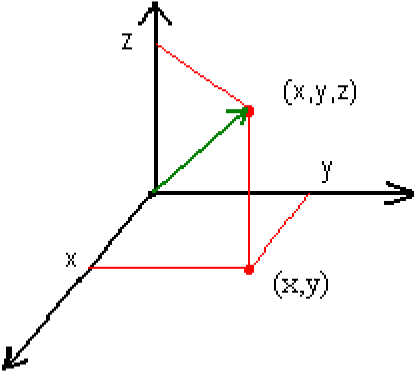
$$\mathbf{v}_1 + \mathbf{v}_2$$

and if $\mathbf{v} \in V$ we define $c\mathbf{v}$ for all $c \in \mathbb{R}$. Elements of $V$ are called *vectors*.

[Additional required properties are below].

# Examples

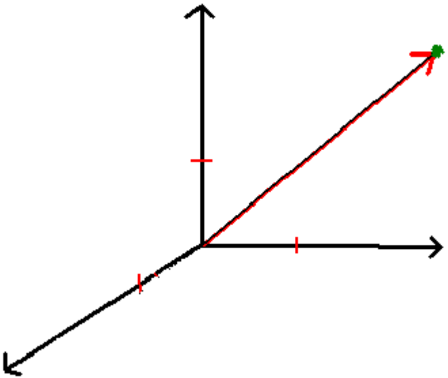
**Ex 1:** $\mathbb{R}^3$:



# Examples

Geometric representation of vector **v:**

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \text{ arrow from origin to } (v_1, v_2, v_3) :$$



## Examples

Addition:

$$\langle 1, 2, 3 \rangle + \langle 2, 1, -1 \rangle = \langle 3, 3, 2 \rangle$$

Scalar multiplication:

$$-2 \langle 1, 2, 3 \rangle = \langle -2, -4, -6 \rangle$$

# Vector space properties

Properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

etc -

# Vector space properties

3. There exists a unique vector **0** $( = \langle 0,0,0 \rangle)$ such that

   $\mathbf{v} + 0 = \mathbf{v}$ for all vectors **v**.

4. For each $\mathbf{v} \in V$, there exists a unique element $-\mathbf{v} \in V$ such that $\mathbf{v} + -\mathbf{v} = 0$
5. for $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$ $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. for $c, d \in \mathbb{R}$ and $\mathbf{u} \in V$, $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for $c, d,$ **u** as above
8. for $\mathbf{u} \in V$, $1 \cdot \mathbf{u} = \mathbf{u}$

Remark on 4: clear if $\mathbf{v} = (v_1, \ldots, v_n)$ then $-\mathbf{v} = (-v_1, \ldots, -v_n)$ in special case $V = \mathbb{R}^n$.

# Vector space properties

**Def. (part II - full definition):** A vector space is a set of objects $V$ on which addition and scalar multiplication are defined with properties 1 - 8 above.

# Examples

**Ex. 2:** $\mathbb{R}^4$, e.g. $\langle x_1, x_2, x_3, x_4 \rangle$ viewed geometrically (arrows) or in terms of vector algebra.

**Ex 3:** $V =$ all polynomials of degree $\leq 2$ form a vector space:

$$\mathbf{v} = a_0 + a_1 x + a_2 x^2$$

# Examples

For example we have for

$$\mathbf{v}_1 = 3 + 2x - 4x^2; \qquad \mathbf{v}_2 = 2 - x + 3x^2$$

$$\mathbf{v}_1 + \mathbf{v}_2 = 5 + x - x^2$$

$$3\mathbf{v}_1 = 9 + 6x - 12x^2.$$

Properties 1 - 8 apply, e.g.:

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$$

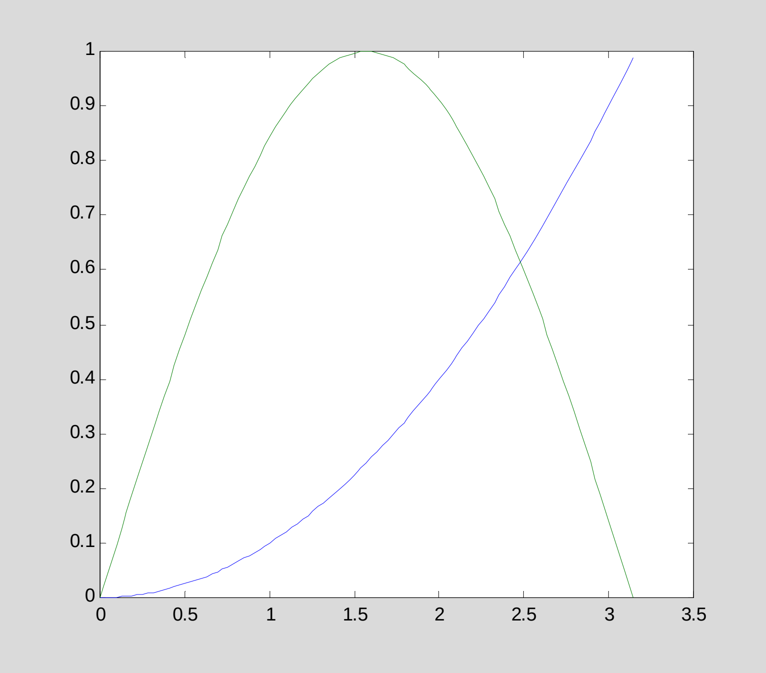
$$4(\mathbf{v}_1 + \mathbf{v}_2) = 4\mathbf{v}_1 + 4\mathbf{v}_2.$$

# Examples

**Ex 4:** $\boldsymbol{V} =$ continuous functions $f(x)$ on $[0, \pi] \subset \mathbb{R}$

$$v_1 = \sin x; \qquad v_2 = x^2/10$$

# Examples



# Examples

Add vectors:

$$\mathbf{v}_1 + \mathbf{v}_2 = \sin x + x^2/10$$

Scalar multiply:

$$3\mathbf{v}_1 = 3\sin x$$

Check vector properties (1-8) satisfied.

[vector space can be: arrows, polynomials, continuous functions; need only define addition, scalar mult, and vector properties (1-8) must hold].

# Properties

## 2. Vector space properties:

**Def 1:** Given vector space $V$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n \in V$, a vector **w** is a *linear combination* (LC) of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$$

for some collection $\{c_1, \ldots, c_n\} \subset \mathbb{R}$.

# Properties

**Ex 5:** $\mathbf{v}_1 = \langle 1, 2, 3 \rangle$; $\mathbf{v}_2 = \langle 2, -1, -1 \rangle$; $\mathbf{v}_3 = \langle -1, 2, 1 \rangle$;

$$\mathbf{w} = \langle -5, 8, 7 \rangle$$

is linear combination of $\mathbf{v}_i$ : can let $c_1 = 1$; $c_2 = -2$; $c_3 = 2$ above.

# Properties

**Ex 6:** $\mathbf{v}_1 = \sin x + 2x^2$; $\mathbf{v}_2 = \sin x - e^x$; $\mathbf{v}_3 = e^x$

$\mathbf{w} = 3\sin x + 6x^2 - 2e^x$ is LC of the $\mathbf{v}_i$, since

$$\mathbf{w} = 3\mathbf{v}_1 + 0\mathbf{v}_2 - 2\mathbf{v}_3$$

**Def 2:** Given vector space $V$ the collection $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} \subset V$ is *linearly independent* (LI) if no $\mathbf{v}_i$ is a LC of the others.

# Properties

Note: if $\exists$ constants $c_1, \ldots c_n$ (not all $0$) s.t.

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = 0,$$

then say if $c_1 \neq 0$,

$$v_1 = -c_2/c_1\, v_1 - \ldots - c_n/c_1 v_n,$$

so one vector is a LC of others and so not lin. ind. Can reverse to show:

**Theorem:** $\mathbf{v}_1 \ldots \mathbf{v}_n$ *are lin. ind. iff do not exist const.* $c_1, \ldots, c_n$ *(some non-zero) s.t.*

$$\mathbf{c}_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = 0.$$

## Properties

**Ex 7:** $\mathbf{v}_1 = \langle 1,2,3 \rangle$; $\mathbf{v}_2 = \langle 2, -1, -1 \rangle$; $\mathbf{v}_3 = \langle -1,2,1 \rangle$ are LI

Reason: if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$,

$$c_1\langle 1,2,3 \rangle + c_2\langle 2,-1,-1 \rangle + c_3\langle -1,2,1 \rangle = \langle 0,0,0 \rangle$$

$$\Rightarrow \quad \begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ 2c_1 - c_2 + 2c_3 &= 0 \\ 3c_1 - c_2 + c_3 &= 0 \end{aligned}$$

$\Rightarrow$ solving, $c_1 = c_2 = c_3 = 0$.

# Properties

**Ex 8:** $\mathbf{v}_1 = \sin x + 2x^2;\ \mathbf{v}_2 = 2\sin x - e^x;\ \mathbf{v}_3 = e^x$ are LI

Proof: $\sum_i c_i \mathbf{v}_i = 0$ holds only with $c_i$ all 0; plug in several $x$ values to show this, e.g., set $x = 0,\ \pi,\ \pi/2$ in the equation $\sum_i c_i \mathbf{v}_i = 0$ to obtain 3 separate equations from which it follows all $c_i = 0$.

[note also that one of the vectors would have to be a combination of the other two for all $x$; plausible that this cannot happen]

## 3. Dimension of a vector space:

Given a vector space $V$

**Def 3:** We say that the set of vectors $S = \{v_1, \dots v_n\}$ *span* $V$ if every vector $w \in V$ can be written as a linear combination of vectors in $S$.

# Basis and dimesion

**Ex 9:** $v_1 = \langle 1, 2, 3 \rangle$; $v_2 = \langle 2, -1, 1 \rangle$; $v_3 = \langle -1, 2, -1 \rangle$ span $V = \mathbb{R}^3$. To see this, note that we have to show that for any vector $w = \langle w_1, w_2, w_3 \rangle$, we can find constants $c_i$ such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = w.$$

[Again involves solving a system of equations; see exercises]

# Basis and dimesion

**Def. 4:** The set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a *basis* for $V$ if

(i) $S$ spans $V$ (i.e., any vector $w \in V$ can be written as a linear combination of vectors from $S$), and

(ii) the vectors from $S$ are linearly independent.

**Theorem 1:** *A set $S \subset V$ is a basis for $V$ iff every $\mathbf{w} \in V$ can be written **uniquely** as a linear combination of vectors in $S$.*

**Ex 10:** $\mathbf{v}_1 = \langle 1, 2, 3 \rangle$; $\mathbf{v}_2 = \langle 2, -1, -1 \rangle$; $\mathbf{v}_3 = \langle -1, 2, 1 \rangle$ are a basis for $\mathbb{R}^3$.

## Basis and dimesion

Every $\mathbf{v} \in \mathbb{R}^3$ can be written as a LC of the $\mathbf{v}_i$ uniquely.

For example

$$\mathbf{w} \equiv \langle -3, 7, 6 \rangle = 1 \cdot \mathbf{v}_1 - 1 \cdot \mathbf{v}_2 + 2 \cdot \mathbf{v}_3,$$

is *only* LC of the $\mathbf{v}_i$ giving $\mathbf{w}$.

# Basis and dimesion

**Ex 11:** $\mathbf{v}_1 = \sin x + 2x^2$; $\mathbf{v}_2 = 2\sin x - e^x$; $\mathbf{v}_3 = e^x$ are a basis for the vector space

$V =$ all linear combinations of $\sin x$, $x^2$ and $e^x$.
$= \{c_1 \sin x + c_2\, x^2 + c_3 e^x \,:\, c_i \in \mathbb{R}\}$.

**Note:** All bases for vector space $V$ have same number $d$ of elements.

$d =$ *dimension* of $V$.

# Vector norms

## 3. Norm (length) of a vector:

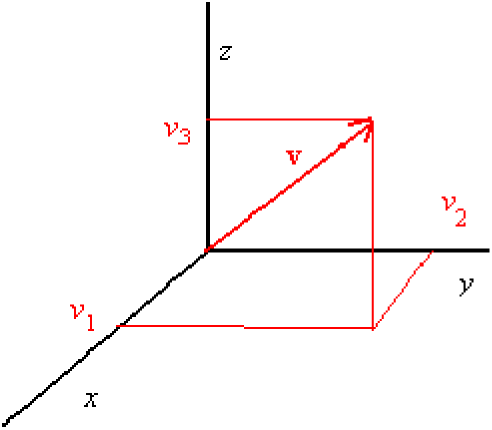
**Notation:** In $\mathbb{R}^3$ write

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

**Def. 5:** Norm $\mathbf{v} = \| \mathbf{v} \|$ distance from end to origin

$$= \sqrt{|v_1|^2 + |v_2|^2 + |v_3|^2}$$

# Vector norms



# 4. Distance between vectors

Between $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$:

$$d = \sqrt{(3-1)^2 + (2-(-2))^2 + (1-1)^2} = \|\mathbf{u} - \mathbf{v}\|^2$$

# Vector norms

Geometry: