Orthogonal vectors

MA 751
Part 2

Inner products

1. Inner product (also known as dot product):

In $\mathbb{R}^n$:

Orthogonal vectors

Inner product of **v** = $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and **w** = $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ is

$$v_1 w_1 + \ldots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

Norm of (real) vector:

$$\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2} = \|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

2. Inner product, geometric:

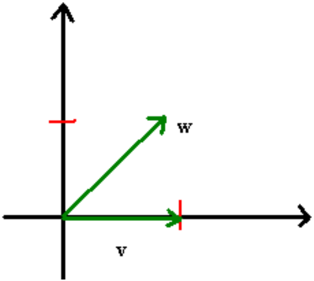
3 D:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$$

where θ = angle between **v** and **w**.

Inner product

Geometry:



$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \; \theta = \pi/4,$$

Inner product is: $\mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 + 0 \cdot 1 = 1$

Inner product

On other hand can use:

$$\|v\|\,\|w\|\cos\theta = 1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 = \mathbf{v} \cdot \mathbf{w}$$

to get same value.

3. Properties of IP:

Theorem 1: For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

(a) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u}$; $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = 0$.
(b) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
(or $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ if $\mathbf{u}, \mathbf{v}$ are complex)
(c) $\langle [\mathbf{u} + \mathbf{v}], \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
(d) $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ if c is a real scalar.

Exercise: Verify these properties for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

Def. 6. $\mathbf{v}$ is a *unit vector* if it has length 1

Inner product

e.g., $\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$ unit vector (after normalization).

4. Inner Product, general:

Can we define IP on abstract vector spaces?

Definition 1: V = vector space. An *inner product* on V is any assignment of a numerical value to $\langle \mathbf{u}, \mathbf{v} \rangle$ which satisfies properties **(a)** to **(d)** of the above theorem.

Any vector space with IP defined is an *inner product space*.

General inner products

Ex 1: Standard IP on $\mathbb{R}^3$;

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i$$

[already know this satisfies **(a)** to **(d)**].

Ex 2: Recall

$$C[-\pi,\pi] = \text{continuous functions on } [-\pi,\pi]$$

are vector space

[note $C[-\pi,\pi]$ has a basis $\{1, \sin nx, \cos nx\}_{n=1}^{\infty}$; infinite dimensional]

General inner products

Define inner product of functions:

$$f \cdot g = \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

[satisfies (a)-(d); check this]

Property **(a)** $(f, f) \geq 0;\ (f, f) = 0$ ***iff*** $f = 0$.

General inner products

Pf: We have

$$(f, f) = \int_0^1 f^2(x)\, dx \geq 0$$

since f^2 is non-negative. Also, if $(f, f) = 0$, then $\int_0^1 f^2\, dx = 0$. From calculus, if a function is nonnegative and its integral is 0, then $f(x) = 0$, as desired. Also, if $f = 0$ then clearly $(f, f) = 0$. $\square$

Other facts proved from the definitions the same way.

General inner products

3. Schwarz inequality and triangle inequality:

Theorem 2 (Cauchy-Schwarz inequality): $|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$ *for any two vectors in a vector space with an inner product.*

Proof: Standard in linear algebra.

General inner products

Ex 3: $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$; then $\mathbf{u} \cdot \mathbf{v} = -2$, while

$$\|\mathbf{u}\| \, \|\mathbf{v}\| = \sqrt{6}\sqrt{9} = 3\sqrt{6} \geq |-2|,$$

as desired.

Theorem 3 (Triangle inequality): *For any two vectors u and v, we have* $\|u + v\| \leq \| v \|$ + $\| v \|$.

Proof: standard in linear algebra.

4. Orthogonal vectors:

Three dimensions: vectors $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$

have zero dot product.

General inner products

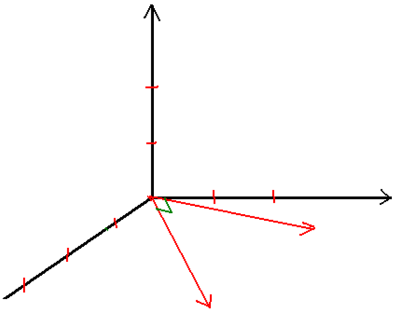


fig 4

Thus we know that $\| \mathbf{v} \| \; \| \mathbf{w} \| \; \cos\theta = 0 \Rightarrow \theta = \pi/2$.

Thus the vectors are perpendicular or *orthogonal.*

Orthogonality

More generally, what happens with higher dimensional vectors?

Note $\begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ also have zero dot product.

So they are perpendicular

Orthogonality

Def 2: We define to vectors **v** and **w** to be *perpendicular* or *orthogonal* if

$$\mathbf{v} \cdot \mathbf{w} = 0.$$

[note this implies angle between them is $\theta = \pi/2$]

Def 3: A collection $S = \{v_1, v_2, \ldots, v_n\}$ of vectors is *orthogonal* if each pair of the vectors ais orthogonal. A collection is *orthonormal* if they are orthogonal and all are unit (length 1) vectors.

Orthogonality

Ex 4:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are orthogonal, not orthonormal.

Orthogonality

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},\ \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

are orthonormal.

[just check dot products and lengths]

Orthogonality

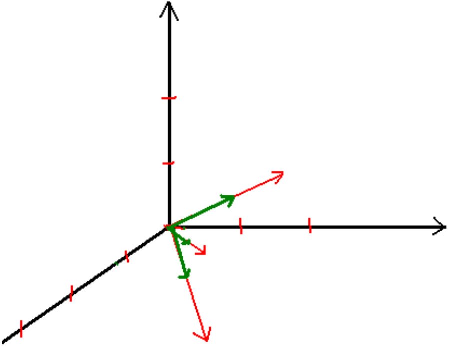


fig 5

Orthonormal bases

3. Advantage of orthonormal bases:

Given an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, how do we express a vector **w** in terms of the vectors in it?

Ex 5: Assume that

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

assume $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Orthonormal bases

Notice that:

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Then:

$$(\mathbf{w}, \mathbf{v}_1) = c_1 = \frac{4}{\sqrt{3}}$$

Orthonormal bases

Similarly

$$(\mathbf{w}, \mathbf{v}_2) = c_2 = \frac{1}{\sqrt{6}}$$

$$(\mathbf{w}, \mathbf{v}_3) = c_3 = -\frac{1}{\sqrt{2}}.$$

Thus expansion is easy to get with orthonormal bases!

[Orthonormal bases make such computations easy].