MA 751
Part 4

Measurability and Hilbert Spaces

1. Measurable functions and integrals

Let C be the set of continuous functions on $\mathbb{R}$. Let M be the set of measurable functions:

Def: The set M of *measurable functions* on $\mathbb{R}$ (or an interval of $\mathbb{R}$) is the set of functions that are limits of continuous functions, i.e.

Measureable functions

$M = \{f(x) : f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x)$ is continuous,

for all $x \in \mathbb{R}\}$.

Measureable functions

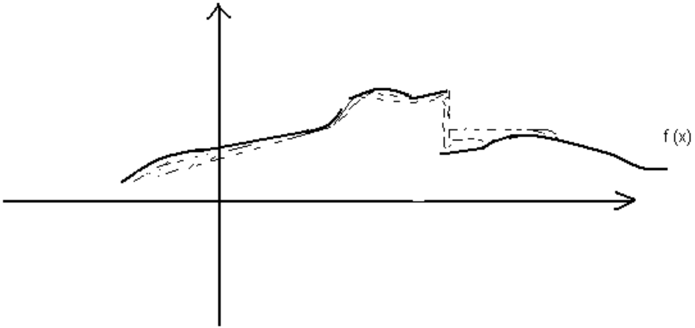

Fig. 1: the function $f(x)$ as a limit of continuous functions

In fact, lots of functions (even discontinuous ones) can be viewed as limits of continuous functions.

Measureable functions

For example

$$f(x) = \chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

is a discontinuous but measurable function.

Note: ordinary notion of integral is difficult to use for functions as complicated as measurable functions.

To integrate measurable functions (Lebesgue integral):

Measureable functions

Theorem: Given a non-negative measurable function $f : \mathbb{R}^p \longrightarrow \mathbb{R}$, there is always an *increasing* sequence $\{f_n(\mathbf{x})\}_{n=1}^{\infty}$ of continuous functions (i.e. with the property that $f_{n+1}(\mathbf{x}) \geq f_n(\mathbf{x})$ for all **x**) which converges to $f(\mathbf{x})$.

Def.: If $f(\mathbf{x}) \geq 0$ is a positive measurable function, define

$$\int_{\mathbb{R}^p} f(x)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^p} f_n(x)\, dx,$$

where $f_n(x)$ is any increasing sequence of continuous functions which converges to f.

Measureable functions

[note we know the value of the integrals of the continuous functions $f_n(\mathbf{x})$ - they are ordinary Riemann integrals on $\mathbb{R}^p$]

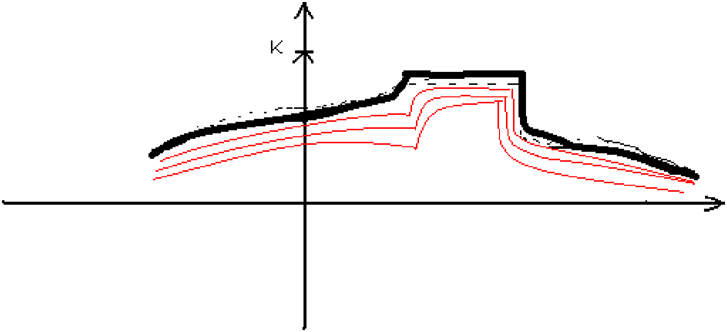

Fig. 2: sequence of continuous functions $f_n(\mathbf{x})$ increasing to $f(\mathbf{x})$

Measureable functions

Def: To find the integral of a negative measurable function f, we just compute the integral of $-f$ (which is positive), and put a minus sign in front of it. Since every function f is the sum of a positive plus a negative function

$$f = f_1 + f_2,$$

the integral of f is defined as

$$\int_{-\infty}^{\infty} f\,dx = \int_{-\infty}^{\infty} f_1\,dx + \int_{-\infty}^{\infty} f_2\,dx.$$

[Thus we now know how to define the integral of an arbitrary function]

Measureable functions

Ex 5: if $f(x)$ looks like:

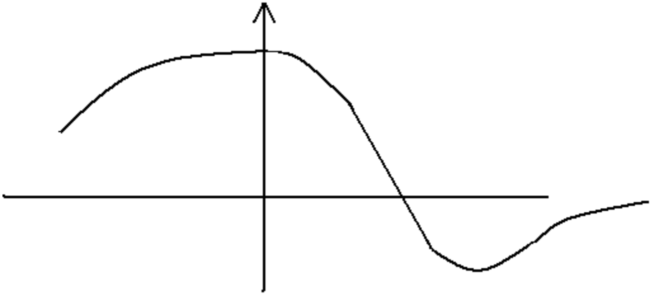

fig 3: $f(x)$ has positive and negative part

Then integral of $f(x)$ is integral of a positive plus a negative function:

Measureable functions

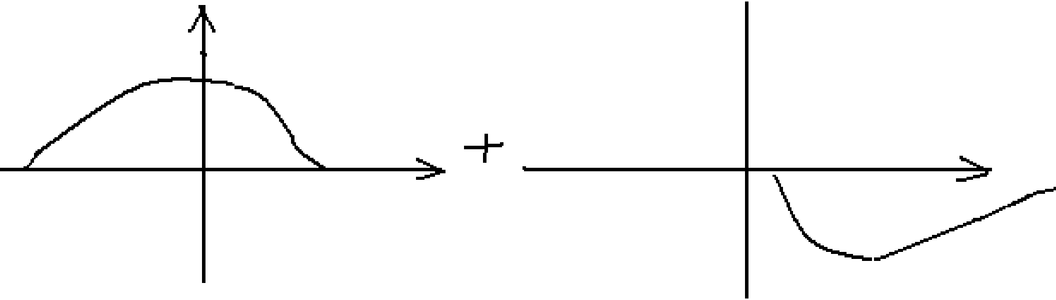

fig 4: now sum the areas between f_1 (or f_2) and the x axis

Measureable functions

Note we can show pretty easily all the properties of integrals we are used to also hold for this more general *Lebesgue integral.* For example, we still have

$$\int (f+g)\, dx = \int f\, dx + \int g\, dx, \text{ etc.}$$

[For now we will assume the above fact.]

2. New Hilbert spaces:

Consider the space

$H = L^2[-\pi,\pi] = \{$measurable real functions $f(x)$ on $[-\pi,\pi]$ with $\int_{-\pi}^{\pi} f^2(x)dx < \infty\}$.

Can show that if $f, g \in H$ then $f+g$ and cf are in H if c is a constant (exercise). More generally H is a vector space.

Further, we can define an inner product on H (known as the L^2 inner product):

Hilbert spaces of functions

$$\langle f, g\rangle = \langle f, g\rangle_{L^2} = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

This satisfies conditions (1) - (4) of an inner product.

Can also show that H is complete (i.e., every Cauchy sequence $\{f_n\}$ converges to a function f in H).

Thus H is a Hilbert space.

Note: we always consider two measurable functions the same if they differ just at a finite number of points

Hilbert spaces of functions

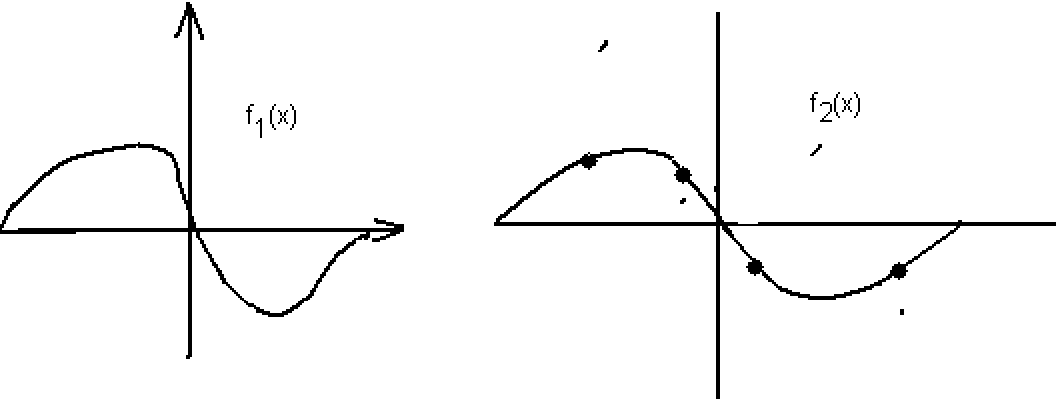

fig 5: two functions f_1 and f_2 which differ at a finite collection of points.

Hilbert spaces of functions

Can show: such functions f_1 and f_2 have the same integral [certainly area is unchanged]; equivalently,

(1) $$\int |f_1 - f_2|\, dx = 0$$

Def 6: More generally we will consider two functions to be the same or *equivalent* if (1) holds

Function space basis expansions

Ex 6: Let $H = L^2[-\pi,\pi]$, with its usual inner product $\langle f,g\rangle = \langle f,g\rangle_{L^2} = \int_{-\pi}^{\pi} f(x)g(x)dx$. Consider the set of vectors

$$B = \{\{\sin nx |\, n = 1, 2, \ldots\} \text{ together with } \{\cos nx |\, n = 0, 1, 2, \ldots$$

$$= \{1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots\}$$

We will show this is an orthogonal set. First: show that 1 is orthogonal to all other vectors:

$$\langle 1,\ \cos nx\rangle = \int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \forall\ n = 1, 2, \ldots$$

Function space basis expansions

$$\langle 1, \sin nx \rangle = \int_{-\pi}^{\pi} \sin nx \, dx = 0 \quad \forall \; n = 1, 2, \ldots$$

Now show that (for example) cos $5x$ is orthogonal to all other vectors:

$$\langle \cos 5x, \sin \; nx \rangle = \int_{-\pi}^{\pi} \cos 5x \sin \; nx = 0 \;\; \forall \quad n = 1, 2, \ldots$$

Function space basis expansions

To show above we use the trig identities:

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

and

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\sin a \sin b = -\frac{1}{2}[\cos(a+b) - \cos(a-b)].$$

Function space basis expansions

[Similarly for any other cos mx.]

$$\langle \cos 5x,\, \cos nx \rangle = \int_{-\pi}^{\pi} \cos 5x \cos nx \, dx = 0 \;\; \forall \;\; n \neq 5$$

Can similarly show that sin mx is also orthogonal to all other vectors.

Thus these vectors form a orthogonal set of vectors. Are they orthonormal?

$$\|\cos nx\|^2 = (\cos nx,\, \cos\, nx) = \int_{-\pi}^{\pi} \cos^2 nx dx$$

$$= \int_{-\pi}^{\pi} \frac{1+\cos\, 2nx}{2} dx$$

$$= \pi$$

Function space basis expansions

Thus:

$$\|\cos nx\| = \sqrt{\pi}.$$

Thus $\frac{1}{\sqrt{\pi}}$ cos nx has length 1.

Similarly, $\frac{1}{\sqrt{\pi}}$ sin nx has length 1

And: $\frac{1}{\sqrt{2\pi}} \cdot 1$ has length 1.

Function space basis expansions

Thus:

$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 3x, \frac{1}{\sqrt{\pi}} \sin 3x, \ldots\}$

$= \{v_1, v_2, v_3, \ldots\}$

Are an orthonormal (and hence lin ind) set for the space of cont. functions.

Can show: they are a basis. So any vector $f(x)$ can be written in the form:

Function space basis expansions

$$f(x) = c_1 v_1 + c_2 v_2 + \ldots$$

$$= c_1 \frac{1}{\sqrt{2\pi}} + c_2 \frac{1}{\sqrt{\pi}} \cos x + c_3 \frac{1}{\sqrt{\pi}} \sin x + c_4 \frac{1}{\sqrt{\pi}} \cos 2x$$

$$+ c_5 \frac{1}{\sqrt{\pi}} \sin 2x + \ldots$$

$$= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

[Fourier series of a function]

Function space basis expansions

Notice that

$$c_4 = \left(f(x),\ \frac{1}{\sqrt{\pi}} \cos 2x\right) = \int_{-\pi}^{\pi} f(x)\, \frac{1}{\sqrt{\pi}} \cos 2x\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos 2x\, dx$$

$$\Rightarrow \qquad a_2 = \frac{c_4}{\sqrt{\pi}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x\, dx$$

Generally:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Function space basis expansions

[Now: no need to do advanced calculus for theory of Fourier series!]

Function space basis expansions

Ex: $f(x) = 2x$

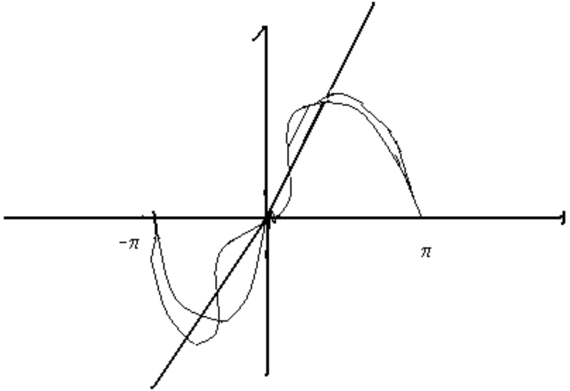

fig 6

Function space basis expansions

$$2x = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

$$b_5 = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \sin 5x \, dx = \frac{2}{\pi} \left\{ - \frac{x \cos 5x}{5} \Big|_{-\pi}^{\pi} + \underbrace{\int_{-\pi}^{\pi} \frac{\cos 5x}{5} dx}_{0} \right.$$

$$= \tfrac{2}{\pi} \left\{ \tfrac{2\pi}{5} \right\} = \tfrac{4}{5}$$

$$b_6 = -\tfrac{4}{6}$$

Function space basis expansions

Generally:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \cos nx = \begin{cases} -\frac{4}{n} & \text{if } n \text{ even} \\ \frac{4}{n} & \text{if } n \text{ odd} \end{cases}$$

Can show $a_n = 0.$

Function space basis expansions

Thus

$$2x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

$$= 4\left[1 \cdot \sin x - \frac{1}{2} \cdot \sin 2x + \frac{1}{3} \cdot \sin 3x + \dots\right]$$

[can draw pictures of first three terms (see earlier); all divided by 2 for the function x]