Measurability and Hilbert Spaces

1. Measurable functions and integrals

Let $C$ be the set of continuous functions on $\mathbb{R}$. Let $M$ be the set of measurable functions:

**Def:** The set $M$ of measurable functions on $\mathbb{R}$ (or an interval of $\mathbb{R}$) is the set of functions that are limits of continuous functions, i.e.
Measureable functions

\[ M = \{ f(x) : f(x) = \lim_{n \to \infty} f_n(x), \text{ where } f_n(x) \text{ is continuous,} \}\]

for all \( x \in \mathbb{R} \).
In fact, lots of functions (even discontinuous ones) can be viewed as limits of continuous functions.
Measureable functions

For example

\[ f(x) = \chi_{[0,1]}(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \\
0 & \text{otherwise} 
\end{cases}. \]

is a discontinuous but measurable function.

Note: ordinary notion of integral is difficult to use for functions as complicated as measurable functions.

To integrate measurable functions (Lebesgue integral):
Measureable functions

**Theorem:** Given a non-negative measurable function $f : \mathbb{R}^p \to \mathbb{R}$, there is always an *increasing* sequence $\{f_n(x)\}_{n=1}^{\infty}$ of continuous functions (i.e. with the property that $f_{n+1}(x) \geq f_n(x)$ for all $x$) which converges to $f(x)$.

**Def.:** If $f(x) \geq 0$ is a positive measurable function, define

$$
\int_{\mathbb{R}^p} f(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^p} f_n(x) \, dx,
$$

where $f_n(x)$ is any increasing sequence of continuous functions which converges to $f$. 
Measureable functions

[Note we know the value of the integrals of the continuous functions $f_n(x)$ - they are ordinary Riemann integrals on $\mathbb{R}^p$]

Fig. 2: sequence of continuous functions $f_n(x)$ increasing to $f(x)$
Measureable functions

Def: To find the integral of a negative measurable function $f$, we just compute the integral of $-f$ (which is positive), and put a minus sign in front of it. Since every function $f$ is the sum of a positive plus a negative function

$$f = f_1 + f_2,$$

the integral of $f$ is defined as

$$\int_{-\infty}^{\infty} f \, dx = \int_{-\infty}^{\infty} f_1 \, dx + \int_{-\infty}^{\infty} f_2 \, dx.$$

[Thus we now know how to define the integral of an arbitrary function]
Measureable functions

Ex 5: if $f(x)$ looks like:

fig 3: $f(x)$ has positive and negative part

Then integral of $f(x)$ is integral of a positive plus a negative function:
Measureable functions

fig 4: now sum the areas between $f_1$ (or $f_2$) and the x axis
Measureable functions

Note we can show pretty easily all the properties of integrals we are used to also hold for this more general Lebesgue integral. For example, we still have

$$\int (f + g) \, dx = \int f \, dx + \int g \, dx, \text{ etc.}$$

[For now we will assume the above fact.]
Hilbert spaces of functions

2. New Hilbert spaces:

Consider the space

\[ H = L^2[-\pi, \pi] = \{ \text{measurable real functions } f(x) \text{ on } [-\pi, \pi] \text{ with } \int_{-\pi}^{\pi} f^2(x) \, dx < \infty \}. \]

Can show that if \( f, g \in H \) then \( f + g \) and \( cf \) are in \( H \) if \( c \) is a constant (exercise). More generally \( H \) is a vector space.

Further, we can define an inner product on \( H \) (known as the \( L^2 \) inner product):
Hilbert spaces of functions

\[ \langle f, g \rangle = \langle f, g \rangle_{L^2} = \int_{-\pi}^{\pi} f(x) g(x) \, dx. \]

This satisfies conditions (1) - (4) of an inner product.

Can also show that \( H \) is complete (i.e., every Cauchy sequence \( \{f_n\} \) converges to a function \( f \) in \( H \)).

Thus \( H \) is a Hilbert space.

Note: we always consider two measurable functions the same if they differ just at a finite number of points.
Hilbert spaces of functions

fig 5: two functions $f_1$ and $f_2$ which differ at a finite collection of points.
Hilbert spaces of functions

Can show: such functions $f_1$ and $f_2$ have the same integral
[certainly area is unchanged]; equivalently,

$\int |f_1 - f_2| \, dx = 0$

Def 6: More generally we will consider two functions to be the
same or equivalent if (1) holds
Ex 6: Let $H = L^2[-\pi, \pi]$, with its usual inner product
\[ \langle f, g \rangle = \langle f, g \rangle_{L^2} = \int_{-\pi}^{\pi} f(x)g(x)\,dx. \]
Consider the set of vectors
\[ B = \{ \sin nx \mid n = 1, 2, \ldots \} \text{ together with } \{ \cos nx \mid n = 0, 1, 2, \ldots \} \]
\[ = \{ 1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots \} \]
We will show this is an orthogonal set. First: show that 1 is orthogonal to all other vectors:
\[ \langle 1, \cos nx \rangle = \int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \forall \ n = 1, 2, \ldots \]
Function space basis expansions

\[ \langle 1, \sin nx \rangle = \int_{-\pi}^{\pi} \sin nx \, dx = 0 \quad \forall \ n = 1, 2, \ldots \]

Now show that (for example) \( \cos 5x \) is orthogonal to all other vectors:

\[ \langle \cos 5x, \sin nx \rangle = \int_{-\pi}^{\pi} \cos 5x \sin nx \, dx = 0 \quad \forall \ n = 1, 2, \ldots \]
Function space basis expansions

To show above we use the trig identities:

\[ \cos a \cos b = \frac{1}{2}[\cos (a + b) + \cos (a - b)] \]

and

\[ \sin a \cos b = \frac{1}{2}[\sin (a + b) + \sin (a - b)] \]

\[ \sin a \sin b = -\frac{1}{2}[\cos (a + b) - \cos (a - b)]. \]
Function space basis expansions

[Similarly for any other cos $m x$.]

$$\langle \cos 5x, \cos nx \rangle = \int_{-\pi}^{\pi} \cos 5x \cos nx \, dx = 0 \quad \forall \quad n \neq 5$$

Can similarly show that sin $m x$ is also orthogonal to all other vectors.

Thus these vectors form a orthogonal set of vectors. Are they orthonormal?

$$\|\cos nx\|^2 = (\cos nx, \cos nx) = \int_{-\pi}^{\pi} \cos^2 nx \, dx$$

$$= \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2} \, dx$$

$$= \pi$$
Function space basis expansions

Thus:

\[ \| \cos nx \| = \sqrt{\pi}. \]

Thus \( \frac{1}{\sqrt{\pi}} \cos nx \) has length 1.

Similarly, \( \frac{1}{\sqrt{\pi}} \sin nx \) has length 1.

And: \( \frac{1}{\sqrt{2\pi}} \cdot 1 \) has length 1.
Function space basis expansions

Thus:

\[
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 3x, \frac{1}{\sqrt{\pi}} \sin 3x, \ldots \right\}
\]

\[= \{v_1, v_2, v_3, \ldots \}\]

Are an orthonormal (and hence lin ind) set for the space of cont. functions.

Can show: they are a basis. So any vector \( f(x) \) can be written in the form:
Function space basis expansions

\[ f(x) = c_1 v_1 + c_2 v_2 + \ldots \]

\[ = c_1 \frac{1}{\sqrt{2\pi}} + c_2 \frac{1}{\sqrt{\pi}} \cos x + c_3 \frac{1}{\sqrt{\pi}} \sin x + c_4 \frac{1}{\sqrt{\pi}} \cos 2x \]

\[ + c_5 \frac{1}{\sqrt{\pi}} \sin 2x + \ldots \]

\[ = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots \]

[Fourier series of a function]
Function space basis expansions

Notice that
\[
c_4 = (f(x), \frac{1}{\sqrt{\pi}} \cos 2x) = \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{\pi}} \cos 2x \, dx
\]
\[
= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos 2x \, dx
\]

\[\Rightarrow \quad a_2 = \frac{c_4}{\sqrt{\pi}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x \, dx\]

Generally:
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx
\]
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]
Function space basis expansions

[Now: no need to do advanced calculus for theory of Fourier series!]
Function space basis expansions

Ex: $f(x) = 2x$
Function space basis expansions

\[ \begin{align*}
2x &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots \\

b_5 &= \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \sin 5x \, dx = \frac{2}{\pi} \left\{ - \frac{x \cos 5x}{5} \right\}_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos 5x}{5} \, dx \\
\quad &= \frac{2}{\pi} \left\{ \frac{2\pi}{5} \right\} = \frac{4}{5} \\
b_6 &= -\frac{4}{6}
\end{align*} \]
Function space basis expansions

Generally:

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \cos nx = \begin{cases} -\frac{4}{n} & \text{if } n \text{ even} \\ \frac{4}{n} & \text{if } n \text{ odd} \end{cases} \]

Can show \[ a_n = 0. \]
Function space basis expansions

Thus

\[ 2x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots \]

\[ = 4 \left[ 1 \cdot \sin x - \frac{1}{2} \cdot \sin 2x + \frac{1}{3} \cdot \sin 3x + \ldots \right] \]

[can draw pictures of first three terms (see earlier); all divided by 2 for the function x]