Suggestions for Problem Set 5

1. (a) Recall that a function f is defined to be piecewise linear on a region R in $\mathbb{R}^p$ if R consists of a finite disjoint collection $R = \bigcup\limits_i R_i$ of sub-regions R_i such restricted to any R_i, f is a linear function of the form $\boldsymbol{\beta}^T\boldsymbol{x} + \boldsymbol{b}$, with $\boldsymbol{\beta} \in \mathbb{R}^p$ a constant vector.

To prove this it may be worthwhile to first prove:

Lemma: The maximum $m(\boldsymbol{x}) \equiv \max(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$ of two continuous functions on a region $R \subset \mathbb{R}^p$ is continuous.

To prove this, you can show that at any fixed $\boldsymbol{y} \in \mathbb{R}^p$ at which $f_1(\boldsymbol{y}) > f_2(\boldsymbol{y})$, by continuity it follows $f_1(\boldsymbol{x}) > f_2(\boldsymbol{x})$ for $\boldsymbol{x}$ sufficiently close to $\boldsymbol{y}$, and so clearly $m(\boldsymbol{x}) \equiv f_1(\boldsymbol{x})$ near $\boldsymbol{y}$, so m is continuous at the point $\boldsymbol{y}$ (why?). Simiarly show the same holds at any $\boldsymbol{y}$ where $f_1(\boldsymbol{y}) < f_2(\boldsymbol{y})$.

Finally, if $f_1(\boldsymbol{y}) = f_2(\boldsymbol{y})$ at some point $\boldsymbol{y}$, then you can use the ϵ-δ definition of continuity to show m is continuous at $\boldsymbol{y}$. That is, show for any $\epsilon > 0$ there is a $\delta > 0$ such that if $|\boldsymbol{x} - \boldsymbol{y}| < \delta$ then $|m(\boldsymbol{x}) - m(\boldsymbol{y})| < \epsilon$. Do this by first finding a $\delta_1$such that if $|\boldsymbol{x} - \boldsymbol{y}| < \delta_1$ then $|f_1(\boldsymbol{x}) - f_1(\boldsymbol{y})| < \epsilon$, and then a δ_2 such that if $|\boldsymbol{x} - \boldsymbol{y}| < \delta_2$ then $|f_2(\boldsymbol{x}) - f_2(\boldsymbol{y})| < \epsilon$ (why do these exist?).

Then show that if $|\boldsymbol{x} - \boldsymbol{y}| < \min(\delta_1, \delta_2)$ then

$$|m(\boldsymbol{x}) - m(\boldsymbol{y})| \le \max(|f_1(\boldsymbol{x}) - f_1(\boldsymbol{y})|, |f_2(\boldsymbol{x}) - f_2(\boldsymbol{y})|) < \epsilon.$$

Thus you should have found the δ you needed for m, namely $\delta = \min(\delta_1, \delta_2)$. Now why is m continuous?

Lemma: The maximum of two *linear* functions f_1 and f_2 on a region R is piecewise linear and continuous.

To prove this, note that $R = R_1 \cup R_2$, where $R_1 = \{\boldsymbol{x} : f_1(\boldsymbol{x}) \ge f_2(\boldsymbol{x})\}$, and $R_2 = R \sim R_1$. Show that $m(\boldsymbol{x}) = f_1(x)$ in R_1 and $m(\boldsymbol{x}) = f_2(x)$ in R_2, and hence that $m(\boldsymbol{x})$ is piecewise linear.

To prove the theorem, for f_1, f_2 piecewise linear you can consider the collection of regions R_i with $\bigcup\limits_i R_i = R$ such that f_1 is linear on each R_i (why does this exist?), and the collection S_j with $\bigcup\limits_j S_j$ such that f_2 is linear on each S_j, so that each f_i is piecewise linear. Why is $m(x)$ linear on each domain $S_j \cap R_i$? How does that prove the result?

(b) To show $f_1 + f_2$ is continuous piecewise linear, try a similar argument

(c) To show $f_2(f_1(x))$ is continuous linear, this time consider the collection R_i of regions on which f_1 is linear, and the collection S_j of regions on which f_2 is linear. Note that here each S_j is just a line segment, since we assume f_1 maps into $\mathbb{R}$, so the domain of f_2 is $\mathbb{R}$ (why?).

Try to show that for each $S_j \subset \mathbb{R}$, the set $f_2^{-1}(S_j) = \{\boldsymbol{x} : f_2(\boldsymbol{x}) \in S_j\}$ is a region T_j in $\mathbb{R}^p$. Show then that on the each nonempty set $R_i \cap T_j$, the function $f_2(f_1(\boldsymbol{x}))$ is linear. How does that prove piecewise linearity?

Finally, you may assume the result from calculus which says that the composition of two continuous functions is continuous, but please state this (known) fact precisely.

2. For a simple introduction to the use of Tensorflow, take a look at https://www.tensorflow.org/tutorials?authuser=1