

Because the data assignment is due on Tuesday April 5, the final due date for this problem set is on Friday April 8, at 4 pm.

### Lectures 15, 16

Reading: Lecture notes, 5.8

The material in this problem set starts with elementary notions in linear algebra that we will need for the discussion of reproducing kernel Hilbert spaces. The basics of this topic are not handled thoroughly in the textbook, but are covered by the class discussion and the initial problems.

It then covers the extended discussion we have had involving functional analysis and reproducing kernel Hilbert spaces. The problems then complete some of this discussion.

**1. (a)** Prove the functions  $v_1 = \sin x + 2x^2$ ;  $v_2 = 2 \sin x - e^x$ ;  $v_3 = e^x$  are linearly independent as elements of the vector space of all continuous functions on the real line  $\mathbb{R}$ .

**(b)** Prove this set of vectors is a basis for the vector space  $V$  which consists of all linear combinations of the functions  $\sin x$ ,  $x^2$ , and  $e^x$  defined for real  $x$ .

**2.** For two continuous real-valued functions  $f$  and  $g$  defined on  $[0,1]$ , show that the definition  $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$  satisfies all the properties of an inner product.

**3.** Let  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be an orthonormal basis for the vector space  $V$ . Show that any vector  $\mathbf{w} \in V$  can be written

$$\mathbf{w} = c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n, \quad \text{with} \quad c_i = \langle \mathbf{v}_i, \mathbf{w} \rangle.$$

**4. Completeness:** Show that the vector space  $H = \mathbb{R}^3 = \{\mathbf{v} = (v_1, v_2, v_3) | v_i \in \mathbb{R}\}$  with inner product defined by  $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^3 u_i v_i$  is complete. Here  $\mathbf{u} = (u_1, u_2, u_3)$ ;  $\mathbf{v} = (v_1, v_2, v_3)$ . You may assume that the real numbers are complete.

**5. Fourier series:** Find the Fourier series (on the interval  $(-\pi, \pi)$ ) of the function .

$$f(x) = \begin{cases} 1 & \text{if } -\pi \leq x < 0 \\ -1 & \text{if } 0 \leq x \leq \pi \end{cases}.$$

**6. The square integrable functions form a Hilbert space:** Here we will try to establish that if  $A$  is a compact (i.e., closed and bounded) set in  $\mathbb{R}^p$ , then

$L^2(A) = \{\text{measurable functions } f(\mathbf{x}) \text{ defined on } \mathbf{x} \in A \text{ such that } \int_A f^2(\mathbf{x})d\mathbf{x} < \infty\}$  is a Hilbert space.

(a) Let  $A$  be a compact set. Show that

$H = L^2(A) = \{\text{measurable functions } f(x) \text{ on } A \text{ with } \int_A f^2(\mathbf{x})d\mathbf{x} < \infty\}$  is a vector space. (You may prove or assume that sums and products of measurable functions are measurable.)

(b) Show that if  $A$  is a compact set in  $\mathbb{R}^p$ , the inner product of two functions  $f$  and  $g$  on  $A$ , defined by

$$\langle f, g \rangle = \int_A f(\mathbf{x})g(\mathbf{x})d\mathbf{x},$$

satisfies the four properties of an inner product. You may assume the basic fact of measure theory that a non-negative measurable function  $f$  which satisfies  $\int_A f(\mathbf{x})d\mathbf{x} = 0$  must be the zero function, and may use the fact that the basic properties of integrals hold for integration of measurable functions (e.g., integral of a sum is a sum of integrals, etc.).

This will show that  $L^2(A)$  is an inner product space. The fact that it is complete (and thus a Hilbert space) can also be shown, though we will not do this here.

7. A reproducing kernel Hilbert space  $\mathcal{H}$  has a reproducing kernel  $K(\cdot, \cdot)$ . For

$$g(\mathbf{x}) = \sum_{i=1}^n c_i K(\mathbf{x}, \mathbf{x}_i) \text{ and } h(\mathbf{x}) = \sum_{i=1}^n d_i K(\mathbf{x}, \mathbf{x}_i), \text{ find } \|g\|_{\mathcal{H}}, \|h\|_{\mathcal{H}} \text{ and } \langle g, h \rangle_{\mathcal{H}}.$$