**MA 751**
**M. Kon**

**Problem Set 9**
**Due Thurs. 4/14/22**

**Lectures 17, 18**

Reading: Lecture notes, 5.8, 7.9, 7.10, 7.11

This material largely covers the extended discussion we have had involving functional analysis and reproducing kernel Hilbert spaces. The problems complete some of this discussion.

**1. (More on RHKS)** Consider the Hilbert space $L^2(F)$ of square integrable functions where $F \subset \mathbb{R}^d$ is compact (i.e. closed and bounded), together with an orthonormal basis $\{\phi_k(\mathbf{x})\}_{k=1}^{\infty}$ of uniformly bounded functions (i.e. $|\phi_k(\mathbf{x})| \le M$ for some fixed $M$ and for all $k$, $\mathbf{x}$). Consider now the sub-space $\mathcal{H}$ consisting of functions $f(\mathbf{x}) = \sum_{k=1}^{\infty} a_k \phi_k(\mathbf{x}) \in L^2(F)$, with inner product

$$f, g\rangle_{\mathcal{H}} \equiv \sum_{k=1}^{\infty} \frac{a_k b_k}{\gamma_k} \ .$$

if $g = \sum_{k=1}^{\infty} b_k \phi_k(\mathbf{x})$. A function $f \in \mathcal{H}$ iff $||f||_{\mathcal{H}}^2 = \langle f, f \rangle < \infty$ for $f$ as above , with $\gamma_k > 0$ for all $n$.

**(a)** Prove that $\mathcal{H}$ is a Hilbert space (proving that it is complete is optional).
**(b)** Find conditions under which $\mathcal{H}$ is an RKHS. You can answer this by finding a simple condition on the $\{\gamma_k\}$ which would guarantee $\mathcal{H}$ is an RKHS in the sense of our original definition.
**(c)** Find the reproducing kernel $K(\mathbf{x},\mathbf{y})$ in this case. Show that $K$ has the reproducing property $\langle f(\cdot) K(\cdot, \mathbf{x}) \rangle_{\mathcal{H}} = f(\mathbf{x})$ for all $f \in \mathcal{H}$.

**2.** Hastie, problem 5.15

**3.** Hastie, Problem 5.16.

**4.** Hastie, Problem 7.8.