Suggestions for Problem Set 9

2. (14.5) To prove $d(F, G) = 0$ iff $F = G$, show that if $F(x_0) > G(x_0)$ and $x_0$ is a continuity point of both functions, then $d(F, G)$ cannot be 0. To show the triangle inequality, show first that for any $\epsilon_i$ such that for all $x$, $F(x - \epsilon_1) - \epsilon_1 \leq G(x) \leq F(x + \epsilon_1) + \epsilon_1$ and 

$$G(x - \epsilon_2) - \epsilon_2 \leq H(x) \leq G(x + \epsilon_2) + \epsilon_2,$$

it follows that for all $x$

$$F(x - \epsilon_1 - \epsilon_2) - \epsilon_1 - \epsilon_2 \leq H(x) \leq F(x + \epsilon_1 + \epsilon_2) + \epsilon_1 + \epsilon_2.$$ 

To prove sufficiency we need consider only continuity points $x$ of $F$, and limits commute with $F(x)$ at these points. To prove necessity Billingsley’s suggestion is sufficient.

3 (25.1) 

(b) If $P$ is Lebesgue measure on $[0, 1]$, show 

$$P(f_n \neq 0) = P(d_{n+1} \neq 0) \ldots P(d_{2n} \neq 0) = \frac{1}{2^n}.$$ 

Thus show fixing $N$ on the right and letting it get large,

$$\{x : f_n(x) \text{ does not converge to } 0\} \subset \{x : f_n \neq 0 \text{ for any } n \geq N\} \xrightarrow{N \to \infty} 0$$

Fixing $N$ and given $x_0 = \frac{k}{2^N} \in (0, 1)$, show if $n \geq N$

$$F_n(x_0) = \int_0^{x_0} f_n(x)dx = 2^n P(x : f_n(x) \neq 0 \text{ and } x \leq x_0).$$ 

But show for this $x_0$, events $A = \{x : f_n(x) \neq 0\}$ and $B = \{x : x \leq x_0\}$ are independent (i.e. $P(A \cap B) = P(A)P(B)$). Thus by (2),

$$F_n(x_0) = P(x \leq x_0) = x_0$$

($x_0 = \frac{k}{2^N}$ is still fixed). Now defining $x_0$ and $x_1$ as below, we have

$$x_0 = \frac{k}{2^N} \leq x \leq \frac{k+1}{2^N} = x_1,$$

and

$$x_0 = F_n(x_0) \leq F_n(x) \leq F_n(x_1) = x_1.$$ 

Let $N \to \infty$ (and adjust $k$ so (3) holds), and show $F_n(x) \xrightarrow{n \to \infty} x$, i.e., the d.f. of Lebesgue measure $P$.

(c) Consider the densities $f_n(x) = nI_{[1/2, 1/2+1/n]}(x)$. What distribution $F$ do the corresponding distribution functions $F_n$ converge to?

(d) Consider Lebesgue measure $P$ on $(0, 1]$, and define distribution functions
where the right side is the binary expansion of $x$. Show that $F_n$ has jumps only at points $\frac{k}{2^n}$, and so is discrete. What is the limiting distribution? Why?

(e) Following Billingsley, note that $G$ is an open set of measure $\leq 1/2$ which contains all rational numbers. It can be constructed for example by listing the rationals $r_1, r_2, \ldots$, and defining

$$G = \bigcup_{i=3}^{\infty} (r_i - 1/2^i, r_i + 1/2^i);$$

note $G$ is open (why) and its measure is bounded above by 1/2 (it doesn't matter whether the above intervals overlap - why?). Defining the $f_n$ and $f$ as in Billingsley, show that the limiting distribution function is

$$F(x) = \lim_{n \to \infty} \int_0^x f_n(x) \, dx = \lim_{n \to \infty} \left( \text{sum of areas of triangles before } x \right) = x.$$  

Also show that if $f(x) = F'(x)$, then $\int_G f_n \, dx = 1$, while $\int_G f \, dx \leq 1/2$.

4. (25.3) (b) Note $\log_{10} x_n$ is uniformly distributed modulo 1 if the fractional part $\{\log_{10} x_n\}$ has a uniform distribution on $[0, 1]$. Let $Y_n(\omega)$ be a rv on the same probability space as $x_n(\omega)$, with

$$Y_n(\omega) = \begin{cases} 
1 & \text{if } \log_{10} d \leq \log_{10} x_n \leq \log_{10} (d + 1) \\
0 & \text{otherwise}
\end{cases}$$

Show $N_n(d) = \sum_{k=1}^{n} Y_k$ and use the law of large numbers.

5. (25.13) (a) Let $F_n$ and $F$ be the corresponding distribution functions and assume $\mu_n \Rightarrow \mu$ (equivalently $F_n \Rightarrow F$). Use the definition of $\mu$ in terms of $F$ to show that if $a, b$ are continuity points of $\mu$ (or equivalently of $F$) then

$$\mu_n(a, b) \underset{n \to \infty}{\longrightarrow} \mu(a, b).$$  

Conversely, if (2) holds for continuity points of $F$, continuity points $x_0$ and $x_1$ such that $F(x_0) < \epsilon$ and $F(x_1) > 1 - \epsilon$. Show that for $n$ sufficiently large $\mu_n(x_0, x_1] = F_n(x_1) - F_n(x_0) > 1 - 3\epsilon$. For such $n$, show that

$$F_n(x_0) \leq 1 - (F_n(x_1) - F_n(x_0)) \leq 3\epsilon.$$  

Thus

$$\limsup_{n \to \infty} |F_n(x) - F(x)|$$
\[
\leq \limsup_{n \to \infty} |F_n(x) - F_n(x_0) - (F(x) - F(x_0))| + |F_n(x_0) + F(x_0)| \\
\leq \limsup_{n \to \infty} |\mu_n(x_0, x) - \mu(x_0, x)| + \limsup_{n \to \infty} F_n(x_0) + F(x_0) \\
\leq 4\epsilon.
\]

Conclude that \( \lim_{n \to \infty} F_n(x) = F(x) \), i.e., that \( F_n \) converge weakly to \( F \).

(b) For an interval \((a, b]\) where \( a, b \) are continuity points, consider

\[
f(x) = \begin{cases} 
1 & \text{if } x \in (a, b] \\
0 & \text{if } x > b' \text{ or } x < a' \\
piecewise \text{ linear and continuous otherwise}
\end{cases},
\]

where \( a' < a < b < b' \).

Show

\[
\limsup_n \mu_n(a, b] \leq \limsup_n \int f \, d\mu_n = \lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu \leq \mu(a', b'].
\]

Taking limits of \( a' \) and \( b' \), conclude that

\[
\limsup_n \mu_n(a, b] \leq \mu(a, b].
\]

Now switching the positions of \( a \) and \( a' \) as well as \( b \) and \( b' \) above, show

\[
\liminf_n \mu_n(a, b] \geq \mu(a, b].
\]