Product Measures and Fubini's Theorem

1. Product Measures

Recall: Borel sets $\mathcal{B}^k$ in $\mathbb{R}^k$ are generated by open sets. They are also generated by rectangles $R = J_1 \times \ldots \times J_k$ which are products of intervals $J_i$.

Let $\mathcal{C}$ be the collection of all rectangles - we have shown that $\mathcal{B}^k = \sigma(\mathcal{C})$, i.e. the Borel sets are the smallest $\sigma$-field containing all rectangles in $\mathbb{R}^k$.

Now we distinguish rectangles (products of intervals in $\mathbb{R}^1$) from measurable rectangles (products of arbitrary Borel sets in $\mathbb{R}^1$)

**Definition 1:** The measureable rectangles $\mathcal{D}$ in $\mathbb{R}^k$ are the collection of products of Borel sets:

$$\mathcal{D} = \{ B_1 \times \ldots \times B_k : B_i \text{ are Borel in } \mathbb{R} \}$$

We know that Borel sets are $\sigma(\mathcal{C})$. But we can show they are also $\sigma(\mathcal{D})$.

**Lemma 1:** The Borel sets are the same as $\sigma(\mathcal{D})$.

**Proof:** We need to show that $\sigma(\mathcal{D})$ is contained in Borel sets $\mathcal{B}^k$. If $J_i$ are open intervals, then $J_1 \times J_2 \times \ldots \times J_k \in \mathcal{B}^k$, and further the collection $\{ A : A \times J_2 \times \ldots \times J_k \in \mathcal{B}^k \}$ is easily seen to be a $\sigma$-field (which contains intervals), and so must be $\mathcal{B}$. Thus for $B_1 \in \mathcal{B}$, it follows that $B_1 \times J_2 \times \ldots \times J_k \in \mathcal{B}^k$ if $J_i$ are open intervals. A similar argument shows that for $B_2 \in \mathcal{B}$ and $J_i$ are open intervals, then $B_1 \times B_2 \times J_3 \times \ldots \times J_n \in \mathcal{B}^k$.

Continuing this way, we conclude that $B_1 \times \ldots \times B_k \in \mathcal{B}^k$ for Borel $B_i$, showing that $\sigma(\mathcal{D}) \subset \mathcal{B}^k$.

This shows that $\sigma(\mathcal{D}) = \mathcal{B}^k$. \hfill $\square$

Recall Lebesgue measure $m$ on $\mathbb{R}^k$ is defined by extending it from the rectangles $\mathcal{C}$ uniquely. It is now easily shown that $m$ can also be extended
from the measurable rectangles $\mathcal{D}$ uniquely, since for $B_i \in \mathcal{B}$, it is easy to show that
\[ m(B_1 \times \ldots \times B_k) = m(B_1) \ldots m(B_k) \]

Note that products of measures $\mu_i$ on $\mathbb{R}$ can be defined analogously to Lebesgue measure on products of copies of the real line. Specifically, if
\[
\begin{align*}
\mu_1 & \text{ is a Borel measure on } \mathbb{R} \\
\mu_2 & \text{ is a Borel measure on } \mathbb{R} \\
& \vdots \\
\mu_k & \text{ is a Borel measure on } \mathbb{R}
\end{align*}
\]

Then we can define the **product measure** $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_k$ on $\mathbb{R}^k$ as follows:

Recall $\mathbb{R}^k = \{(x_1, \ldots, x_k) : x_i \in \mathbb{R}\}$.

Let 
\[
\begin{align*}
B_1 & \in \mathcal{B} = \text{Borel sets on } \mathbb{R} \\
B_2 & \in \mathcal{B} \\
& \vdots \\
B_k & \in \mathcal{B}
\end{align*}
\]

We can similarly define
\[ \mu(B_1 \times \ldots \times B_k) = \mu_1(B_1) \ldots \mu_k(B_k). \]

and then extend to all Borels $\mathcal{B}^k$.

Thus:

**Theorem 2:** There exists a unique measure $\mu$ on $\mathbb{R}^k$ defined by
\[ \mu(B_1 \times B_2 \times \ldots \times B_k) = \mu_1(B_1)\mu_2(B_2)\ldots\mu_k(B_k) \]
Of course, this definition generalizes to any product of measure spaces

$$(X, \mathcal{X}, \mu) \times (Y, \mathcal{Y}, \nu)$$

$$\Omega = \mathcal{X} \times \mathcal{Y} = \{(x, y) : x \in X, \ y \in Y\}$$

Thus: on $\Omega$ we define a $\sigma$-field $\mathcal{F} = \mathcal{X} \times \mathcal{Y} = \sigma\{A \times B : A \in \mathcal{X}, \ B \in \mathcal{Y}\}$.

Measure $\pi = \mu \times \nu$ as before is defined by

$$m(A \times B) = \mu(A) \times \nu(B).$$

**Theorem 18.1 [Billingsley]:** (a) If $E \in \mathcal{X} \times \mathcal{Y}$, then for each $x$, 
\{y : (x, y) \in E\} \in \mathcal{Y}$, and similarly for each $y$, \{x : (x, y) \in E\} \in \mathcal{X}$.

(b) If $f(x, y)$ measurable wrt $\mathcal{X} \times \mathcal{Y}$, then for each fixed $x$ $f(x, \cdot)$ is measurable with respect to $\mathcal{Y}$, and for each fixed $y$ $f(\cdot, y)$ is measurable with respect to $\mathcal{X}$.

**Proof:** Given fixed $x$ the map $T_x(y) = (x, y)$ maps $Y$ to $X \times Y$. Easy to show that $T_x^{-1}(E)$ is measurable if $E = A \times B$ is a measurable rectangle. Thus by Theorem 13.1 $T$ is measurable since rectangles $E$ generate all measurable sets. Thus for any $E$ we have $T_x^{-1}(E) = \{y : (x, y) \in E\}$ is measurable. The final claim follows easily. $\square$

[simple proof in Billingsley]

Now consider a set $E \in \mathcal{X} \times \mathcal{Y}$. Define $E_x = T_x^{-1}(E)$. Define the function $m(x) = \nu(E_x) = \nu\{y : (x, y) \in E\}$. It is easy to show that $m(x)$ is measurable (see Billingsley). Define

$$\pi'(E) = \int_X \nu(E_x) \, d\mu(x)$$

$$\pi''(E) = \int_Y \mu(E_y) \, d\nu(y)$$

where
Easy to show that \( \pi' \) and \( \pi'' \) are measures. Also easy to show that if \( E = A \times B \) is a measureable rectangle then

\[
\pi(E) = \mu(A)\nu(B) = \pi'(E) = \pi''(E).
\]

Using unique extension (see Billingsley; this must be done first for finite and then \( \sigma \)-finite measure spaces), we obtain:

**Theorem 18.2 [B]:** The measure \( \pi = (\mu \times \nu) \) defined by

\[
\pi(A \times B) = \mu(A)\nu(B)
\]

is \( \sigma \)-finite and coincides with \( \pi' \) and \( \pi'' \) above. It is its own unique extension when restricted to measureable rectangles \( E = A \times B \).

**Example 1:** Lebesgue measure on \( \mathbb{R}^2 \) is \( \mu = \mu_1 \times \mu_2 \) where \( \mu_i \) are Lebesgue measure.

2. **The Fubini theorem (analysis; section 18)**

**Fubini Theorem (section 18):** If \( f(x, y) \) is measurable with respect to \( \mu_1 \times \mu_2 \), and \( \int |f|d(\mu_1 \times \mu_2) < \infty \), then

\[
\int \left( \int f(x, y)d\mu_1(x) \right)d\mu_2(y) = \int f(x, y)d(\mu_1 \times \mu_2)
\]

\[
= \int \left( \int f(x, y)d\mu_2(y) \right)d\mu_1(x).
\]

This is effectively a fundamental theorem of calculus (an evaluation theorem) for multiple integrals.

**Proof:** In the case where \( f(x, y) = I_E(x, y) \) where \( E \in \mathcal{X} \times \mathcal{Y} \), this is easy to show (follows directly from Theorem 18.2 above). On the other
hand, this then extends to simple functions and then by taking limits directly to positive functions, and then all functions. □