Notes on the Unique Extension Theorem

1. More on measures:

Recall that we were interested in defining a general measure of a size of a set on $[0, 1]$.

Defined this measure $P$. Defined

$$P((a, b)) = b - a.$$ 

Question: how large a collection of sets $\mathcal{F}$ can we extend this definition to?

Decided we cannot consistently extend this definition to all subsets of $[0, 1]$.

Goal: find a general probability measure $P$ on $[0, 1]$ with

$$P((a, b)) = b - a$$

defined on a 'reasonably large' collection $\mathcal{F}$ of subsets of $\mathbb{R}$.

Recall we were able to define $P$ on the collection

$$\mathcal{F}_0 = \{ \text{all finite unions of disjoint open intervals } I_i \}$$

$$= \{ \bigcup_{i=1}^{n} I_i | I_i = \text{intervals (can be open, closed, or half-open)} \}$$

We claimed we could extend it to a larger collection of sets

$$\mathcal{F} = \sigma\text{-field generated by } \mathcal{F}_0.$$ 

More generally: We are on a set $\Omega$, where we have a field of sets $\mathcal{F}_0$ on which we have a probability measure $P$. Recall $P$ is a probability measure on a field $\mathcal{F}_0$ of sets if:

(i) $P(\emptyset) = 0$
(ii) $P(\Omega) = 1$
(iii) \( P \) is countably additive on \( \mathcal{F}_0 \) : that is, if \( A_i \) is a sequence of disjoint sets in \( \mathcal{F}_0 \), and if \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_0 \), then \( \sum_{i=1}^{\infty} P(A_i) = P(A) \).

We need:

**Theorem (Unique extension theorem):** Any set function \( P \) defined on a field \( \mathcal{F}_0 \) of sets and satisfying the properties of a probability measure on \( \mathcal{F}_0 \) extends uniquely to a probability measure on the \( \sigma \)-field \( \mathcal{F} \) generated by \( \mathcal{F}_0 \).

2. Outer measures

Assume we have a field \( \mathcal{F}_0 \) (for example finite unions of open sets) of sets on a space \( \Omega \).

Let \( P \) be a measure on \( \mathcal{F}_0 \) (note it does not need to be a probability measure). Let \( \mathcal{F} \) be the \( \sigma \)-field generated by \( \mathcal{F}_0 \).

We will show that there is a unique extension of \( P \) from \( \mathcal{F}_0 \) to \( \mathcal{F} \).

We first define the *outer measure* as an extension of \( P \).

**Definition 1:** Given a measure \( P \) on \( \mathcal{F}_0 \) define its *outer measure* \( P^* \) on all susets of \( \Omega \) by

\[
P^*(A) = \inf \left\{ \sum_{i=1}^{\infty} P(A_i) \mid A \subseteq \bigcup_{i=1}^{\infty} A_i; \ A_i \in \mathcal{F}_0 \right\},
\]

i.e. the smallest sum of measures of a collection of \( \mathcal{F}_0 \) sets containing \( A \).

We can also define the inner measure as one minus the largest sum of measures of a collection of \( \mathcal{F}_0 \) sets contained in \( A^c \):
\[ P_*(A) = \sup \left\{ 1 - \sum_{i=1}^{\infty} P(A_i) \middle| A^c \subset \bigcup_{i=1}^{\infty} A_i; A_i \in \mathcal{F}_0 \right\}, \]

But this is equivalent to the following definition of inner measure:

\[ P_*(A) = 1 - P^*(A^c). \]

Want to extend measure \( P \) on \( \mathcal{F}_0 \) to a collection \( \mathcal{F} \) of as many sets as possible.

How about choosing \( \mathcal{F} \) to be the collection of sets \( A \subset \Omega \) which have the same inner and outer measure, and then define that to be \( P(A) \)?

That is, for all sets \( A \) such that \( P^*(A) = P_*(A) \), i.e.,

\[ P^*(A) + P^*(A^c) = 1 \]

(1)

define such sets to be in \( \mathcal{F} \) and define

\[ P(A) = P^*(A) = P_*(A). \]

Is this collection a \( \sigma \)-field?

This is what we want, but we want to define a more restrictive condition on \( A \) which turns out to be the same.

We will define a set to be \( P^* \)-measurable if (extend condition (1)): for every set \( E \subset \Omega \),

\[ P^*(A \cap E) + P^*(A^c \cap E) = P^*(E). \]

(2a)

Notice that when \( E = \Omega \) then (2) reduces to (1).

Define \( \mathcal{M} \) to be the collection of \( P^* \)-measurable sets.
Claim: the measure $P^*$ on $\mathcal{M}$ is essentially the extension of $P$ on $\mathcal{F}_0$ that we want.

Easy to check:

(a) $P^*$ is monotone, i.e., if $A \subset B$ then $P^*(A) \leq P^*(B)$
(b) $P^*$ is sub-additive, i.e.,

$$P^*\left( \bigcup_i A_i \right) \leq \sum_i P^*(A_i).$$

Thus it is automatically true that

$$P^*(A \cap E) + P^*(A^c \cap E) \geq P^*(E),$$
so that (2) is equivalent to

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E), \quad (2b)$$

3. Some properties of the measure $P^*$ and $\mathcal{M}$

**Lemma 1:** $\mathcal{M}$ is a field

**Proof:** Note it suffices to show that (i) $\mathcal{M}$ is closed under complements (easy) and that (ii) $\mathcal{M}$ is closed under intersections.

Reason: then if $A, B \in \mathcal{M}$, we have

$$A \cup B = (A^c \cap B^c),$$
so that $A \cup B \in \mathcal{M}$ also, and arbitrary finite unions follow.

Now to show $\mathcal{M}$ closed under intersections:

If $A, B \in \mathcal{M}$, then for any set $E \subset \Omega$,

$$P^*(E) = P^*(B \cap E) + P^*(B^c \cap E)$$
\[ = P^*(A \cap B \cap E) + P^*(A^c \cap B \cap E) \]
\[ + P^*(A \cap B^c \cap E) + P^*(A^c \cap B^c \cap E) \]
\[ \geq P^*(A \cap B \cap E) + P^*[A \cap (B^c \cap E)] \]
\[ = P^*[A \cap B \cap E] + P^*[(A \cap B)^c \cap E] \]

which using (2b) implies that \( A \cap B \in \mathcal{M} \), as desired. \( \square \)

**Lemma 2:** If \( \{A_i\}_i \) is a finite or infinite sequence of disjoint sets in \( \mathcal{M} \), then if \( E \subseteq \Omega \),

\[ P^*(E \cap \bigcup_i A_i) = \sum_i P^*(A_i). \]

**Proof:** For the case of two \( A_i \), we have to replace \( E \) by \( E \cap (A_1 \cup A_2) \)

\[ P^*[E \cap (A_1 \cup A_2)] \]
\[ = P^*\{[E \cap (A_1 \cup A_2)] \cap A_1\} + P^*\{[E \cap (A_1 \cup A_2)] \cap A_1^c\} \]
\[ = P^*(E \cap A_1) + P^*(E \cap A_2). \]

For the case of more than two \( A_i \), we proceed by induction writing e.g.

\( (A_1 \cup A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3, \)

and so on.

For a countable sequence of \( A_i \), we have by monotonicity, for all \( n \) and then taking the limit in \( n \):

\[ P^*\left(E \cap \left[ \bigcup_{i=1}^{\infty} A_i \right]\right) \geq P^*\left(E \cap \left[ \bigcup_{i=1}^{n} A_i \right]\right) = \sum_{i=1}^{n} P^*(E \cap A_i) \]

so letting \( n \to \infty \) we have
\[ P^*(E \cap \left[ \bigcup_{i=1}^{\infty} A_i \right]) \geq \sum_{i=1}^{\infty} P^*(E \cap A_i) ; \]  

(3)

Opposite inequality follows by sub-additivity; thus we have equality in (3) above, as desired. □

**Corollary:** The outer measure \( P^* \) is countably additive on \( \mathcal{M} \).

**Proof:** Just let \( E = \Omega \). □

**Lemma 3:** The collection of sets \( \mathcal{M} \) is a σ-field.

**Proof:** For \( A_i \in \mathcal{M} \), we wish to show that \( B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{M} \). It suffices with small changes to assume that the \( A_i \) are disjoint (see text). Letting \( B_n = \bigcup_{i=1}^{n} A_i \), it is clear that \( B_n \in \mathcal{M} \), since \( \mathcal{M} \) is a field.

Thus

\[
P^*(E) = P^*(E \cap B_n) + P^*(E \cap B_n^c) = \sum_{i=1}^{n} P^*(E \cap A_i) + P^*(E \cap B_n^c)
\]

\[
\geq \sum_{i=1}^{n} P^*(E \cap A_i) + P^*(E \cap B^c).
\]

Letting \( n \to \infty \), we get (using countable additivity, proved earlier, for the equality below)

\[
P^*(E) \geq \sum_{i=1}^{\infty} P^*(E \cap A_i) + P^*(E \cap B^c) = P^*(E \cap B) + P^*(E \cap B^c),
\]

which with (2b) above proves that \( B \in \mathcal{M} \). □

**Lemma 4:** \( \mathcal{F}_0 \subset \mathcal{M} \)

**Proof:** We need to show that if \( A \in \mathcal{F}_0 \) and \( E \subset \Omega \), then
\[ P^*(E \cap A) + P^*(E \cap A^c) = P^*(E). \]

Clearly this is true for \( E \in \mathcal{F}_0 \), since this is just finite additivity.

If \( E \notin \mathcal{F}_0 \), we approximate \( E \) by sets \( D_n \in \mathcal{F}_0 \) from above.

Specifically, for \( \epsilon > 0 \), let \( \{A_i\}_{i=1}^n \) be a finite collection of sets in \( \mathcal{F}_0 \) such that
\[
E \subseteq \bigcup_{i=1}^n A_i \equiv D_n,
\]
and
\[
P^*(E) \leq \sum_{i=1}^n P(A_i) - \epsilon.
\]

Then clearly \( C_n \) is an approximation of \( E \) from above. In this case we have that
\[
P^*(D_n \cap A) + P^*(D_n \cap A^c) = P^*(D_n).
\]

Now take limits as \( n \to \infty \). The right side clearly goes to \( P^*(E) \) by the definition of \( D_n \) above, while the left side goes to\( P^*(E \cap A) + P^*(A \cap A^c) \) by a simple approximation argument. \( \Box \)

4. Completion of the proof of unique extension

We now know that \( P^* \) is defined on a \( \sigma \)-field \( \mathcal{M} \) of sets which extends \( \mathcal{F}_0 \) and which satisfy equation (2) above.

It is easy to show that for \( A \in \mathcal{F}_0 \), we have \( P^*(A) = P(A) \).

**Final step:** Let \( \mathcal{F} = \sigma(\mathcal{F}_0) = \sigma \)-field generated by \( \mathcal{F}_0 \). Then we have
\[
\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{M}.
\]

And since \( P^* \) extends \( P \) to a countably additive measure on \( \mathcal{M} \), the restriction of this measure to \( \mathcal{F} \) gives us the desired extension.

5. Uniqueness and the \( \pi\)-\( \lambda \) theorem
Q: How do we show that the extension of $P$ to $P^*$ is unique, i.e. there is no other extension?

Definition 2: A collection $\mathcal{P}$ of subsets of a space $\Omega$ is a $\pi$-system if it is closed under intersections.

Definition 3: A collection $\mathcal{L}$ of subsets of a space $\Omega$ is a $\lambda$-system if

- $(1) \Omega \in \mathcal{L}$
- $(2) \mathcal{L}$ is closed under complements
- $(3) \mathcal{L}$ is closed under countable disjoint unions

It is easy to show that if a class $\mathcal{C}$ of sets is both a $\pi$-system and a $\lambda$-system, then it is a $\sigma$-field.

Unique extension follows from Dynkin's $\pi$-$\lambda$ theorem:

Theorem 1: If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system, and if $\mathcal{P} \subset \mathcal{L}$, then it follows also that $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Proof: The proof follows from establishing the following easy statements:

1. Define $\mathcal{L}_0$ to be the $\pi$-system generated by $\mathcal{P}$. Then $\mathcal{L}_0 \subset \mathcal{L}$, since $\mathcal{L}_0$ is the minimal $\lambda$ system containing $\mathcal{C}$.

   If we can show that $\mathcal{L}_0$ is also a $\pi$-system, then it is a $\sigma$-field and so

   $$\mathcal{P} \subset \sigma(\mathcal{P}) \subset \mathcal{L}_0 \subset \mathcal{L}$$

   and we will show that $\sigma(\mathcal{P}) \subset \mathcal{L}$.

2. For a set $A$, let $\mathcal{L}_A = \{B : A \cap B \in \mathcal{L}_0\}$. Show that if $A \in \mathcal{P}$ then $\mathcal{L}_A$ is a $\lambda$-system.

3. If $A \in \mathcal{P}$, show that $\mathcal{L}_0 \subset \mathcal{L}_A$.

4. Show that if $B \in \mathcal{L}_0$ then $\mathcal{P} \subset \mathcal{L}_B$.

5. Since $\mathcal{L}_B$ is a $\pi$-system, it follows that if $B \in \mathcal{L}_0$ then $\mathcal{L}_0 \subset \mathcal{L}_B$. 
6. Thus if $B, C \in \mathcal{L}_0$ then $B \cap C \in \mathcal{L}_0$

This shows that $\mathcal{L}_0$ is a $\pi$-system, which was what was left to be proved.

**Theorem 2:** If $\mathcal{P}$ is a $\pi$-system and $P_1, P_2$ are probability measures on $\sigma(\mathcal{P})$, and if they agree on $\mathcal{P}$, they also agree on $\sigma(P)$.

**Proof:** Let $\mathcal{L}$ be the class of sets in $\sigma(\mathcal{P})$ such that $P_1(A) = P_2(A)$. Then if $A \in \mathcal{L}$, then $P_1(A^c) = 1 - P_1(A) = 1 - P_2(A) = P_2(A)$, so that $A^c \in \mathcal{L}$. Thus $\mathcal{L}$ is closed under complements.

Can easily also show that $\mathcal{L}$ is a $\lambda$-system. Thus $\mathcal{P} \subset \mathcal{L}$ and $\mathcal{P}$ is a $\pi$-system, so the $\pi$-$\lambda$ theorem gives $\sigma(\mathcal{P}) \subset \mathcal{L}$, as desired. $\square$