Unique Recovery from Edge Information

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***Abstract*—We study the inverse problem of recovering a function f from the nodes (zeroes) of its wavelet transform. The solution also provides an answer to a generalization of the Marr conjecture in wavelet and mathematical vision theory, regarding whether an image is uniquely determined by its edge information. The question has also other forms, including whether nodes of heat and related equation solutions determine their initial conditions. The general Marr problem reduces in a natural way to the moment problem for reconstructing f, using the moment basis on $\mathbb{R}^d$ (Taylor monomials x^α), and its dual basis (derivatives $\delta^{(\alpha)}$ of of the Dirac delta distribution), expanding the wavelet transform in moments of f. If f has exponential decay and the wavelet's derivatives satisfy generic positions for their zeroes, then f can be uniquely recovered. We show this is the strongest statement of its type. For the original Gaussian wavelet unique recovery reduces to genericity of zeroes of so-called Laplace-Hermite polynomials, which is proved in one dimension.**

I. Introduction

The question of whether nodes (zeroes) of a wavelet transform are sufficient to recover the original function f has several areas of application, in addition to its role as a recovery and inverse problem in wavelet theory. These areas include mathematical vision (e.g. the Marr conjecture) and the study of nodes of heat and also more general PDE. Full recovery using smooth wavelets has been shown impossible ([18], [9]) for some non-decaying f; however such functions are unphysical as images. The question for compactly supported and decaying f has remained open [20], [18], [14], [15].

A source of attention to this problem has been in mathematical vision and image compression [24], [14]. It is known that the sparse information in the sketch of an image is sufficient for natural vision systems (e.g. humans) to extract almost all of its salient information [16], [15]. David Marr [16] analyzed natural vision from a neural viewpoint, and developed a detailed theory of edge detection as an image compression and recovery method. The theory of Marr and others has led to a more basic understanding visual recognition in all neural systems. Marr also formulated a mathematical question related to recovery from edge information, essentially asking whether an image can be recovered from its multiscale edges.

Mathematically an image's edges are defined as zero crossings of the second derivative of its blurred versions at multiple scales (see diagram), with edges obtained as lines of maximal contrast. Such edges at a discrete set of scales $\{\sigma_i\}$ form the family of multiscale edges of the image. A standard example of such blurring is Gaussian blurring, with resulting edges

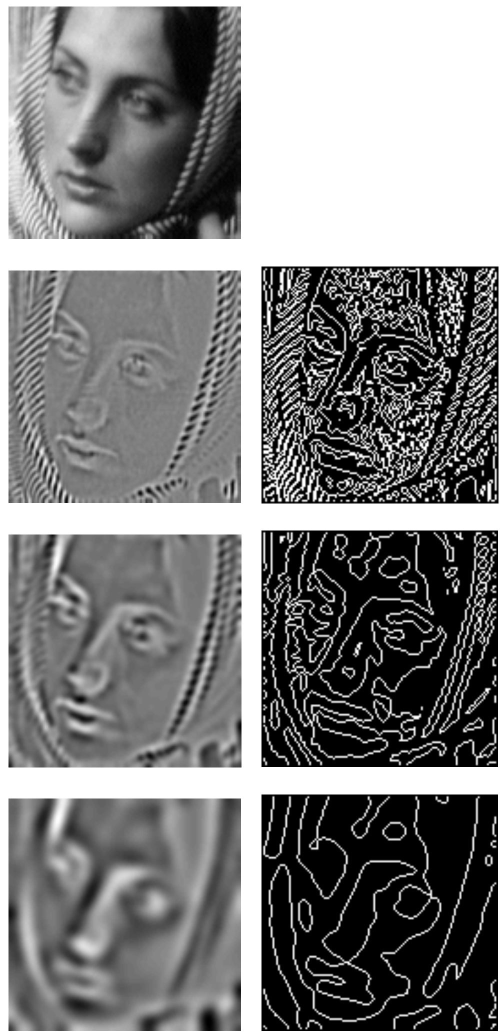

Fig. 1. An image blurred at several scales with the Ricker wavelet $\phi = \Delta G$. This is done line by line using a one dimensional wavelet derivative. Left side: original image, then convolved with different scalings of ϕ. Right side: corresponding edges at increasing scales. https://www.math.ucdavis.edu/∼saito/talks/ucdmathbio.pdf

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known as Gaussian edges.

The mathematical conjecture postulated that knowing the edges of a image uniquely determine it, i.e., that a picture can be recovered from its multiscale edges. The one dimensional version, studied by Mallat [14], [15] in 1989, posits that when a picture is decomposed into horizontal lines, the union of the edges in these lines determine the image. The conjecture has been studied for a period of time, and Jaffard, Meyer and Ryan [18], [9] showed it false for non-smooth wavelets and some images (infinite in extent) that do not decay.

In this announcement we present a solution to a generalized version of the Marr problem, showing that recovering f generally requires that f have exponential decay, and that the blurring wavelet ϕ (assumed smooth) satisfies genericity conditions for the positions of its nodes and those of its derivatives. The proof relies on the moment problem for recovering f. Thus, whether a picture is determined by its multiscale edges depends on the problem of reconstructing f from its moments $\int_{\mathbb{R}^n} x^\alpha f(x)dx$; this in fact would seem to form the natural setting for the mathematical problem.

II. Preliminaries

We consider the wavelet transform of a function f using a wavelet $\tilde{\phi}(x)$, in d dimensions. By convention this is

$$Wf(\sigma, x) = \sigma^{d/2} f(x) * \phi_\sigma(x) \tag{1}$$

where $\phi(x) = \tilde{\phi}(-x)$,

$$\phi_\sigma(x) = \sigma^{-d}\phi(x/\sigma) \tag{2}$$

and $*$ denotes the standard multidimensional convolution

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy. \tag{3}$$

For convenience we also denote ϕ as the wavelet; below all conditions on ϕ will be unchanged by reflection and also apply to $\tilde{\phi}$. We wish to find when f is uniquely determined (up to a constant multiple) by the nodes of its wavelet transform, assuming the integrals above converge absolutely. We study the more general question of whether f can be recovered from the nodes of $Wf(\sigma, x)$ restricted to an arbitrary given scale sequence $\{\sigma_i\}_{i=1}^\infty$. This is an inverse recovery problem for the continuous wavelet transform [14], [15], [18], [9], and also for the dyadic transform [14], [15] (which is continuous in the space variable x but discrete in the scaling σ, generally with $\sigma_i = 2^i$).

In mathematical vision theory [16], convolutions of an image f with a sequence of discrete rescalings ϕ_{σ_i} of $\phi(x) = G(x) = \frac{1}{(2\pi)^{d/2}} e^{-x^2/2}$ form the *Gaussian smoothings* of the image. Letting the Ricker (Gaussian derivative) wavelet $M(x)$ be $\Delta G(x)$ (with Δ the Laplacian), the zeros of $f * M_\sigma(x)$ represent the maximal change points of the blurred images and hence its edges at the scales σ_i. The nodes of $f * M_\sigma(x)$ thus form a sequence of increasingly sparse representations of the image f. In hypoelliptic PDE theory a solution $u(x,t)$ with initial condition $f(x)$ at $t = 0$ f is effectively a convolution of f with a heat or more general kernel of the above form ϕ, and the above question reduces to whether the nodes of such an $u(x,t)$ (generally at a discrete set of times t_i) determine the solution u. There has also been study of this PDE version of the recovery question (partly motivated by the Marr problem), as well as study of the forward problem characterizing nodes of solutions *given* the initial data f [22], [1].

In wavelet theory this has been studied empirically and theoretically by Mallat [14], [15], and Meyer [18], [9]. Mallat showed that, based on a (non-smooth) cubic spline wavelet, it is possible and practical to reconstruct images from multiscale edge information. The theoretical question for Gaussian wavelets has also been studied for special cases and with additional assumptions [3], [20], [17], [16], [8], [24], [2].

The general form of the problem asks for minimal conditions on a wavelet ϕ and a class of functions f that guarantee that a function in this class can be uniquely recovered from its multiscale ϕ-nodes, i.e., from the zeroes of $(\phi)(\sigma^{-n}x) * f(x)$ for a discrete sequence of scales σ_i. This is equivalent to asking whether the zeroes of the convolutions $\phi_{\sigma_i}(x) * f(x)$ determine f. The function f is sometimes called the *initial condition* because of the fact that $f * \phi_\sigma(x)$ solves the heat equation with initial condition f, if we rescale time $t = \sigma^2$.

III. Main Results on unique recovery

For the multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ of non-negative integers α_i, we define $|\alpha| = \sum_i \alpha_i$. We define the partial derivative (here $x = (x_1, \ldots, x_d)$) by

$$\partial^\alpha \phi = \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_d^{\alpha_d}} \phi(x). \tag{4}$$

We will specify the initial assumptions on the function f to be reconstructed, and also the wavelet ϕ. We begin by assuming that f belongs to a class $\mathcal{P}$ of sub-exponential functions ϕ with topology generated by the seminorms (assumed finite) $\|f\|_{\gamma,\alpha} = \sup_x e^{-\gamma|x|}|\partial^\alpha f|$. We assume ϕ belongs to the dual space $\mathcal{P}'$ of distributions growing no more than exponentially. This allows the convolution $f * \phi$ to be defined.

We define a regular (or transverse) zero w of a function $g(w)$ as a zero, in every neighborhood of which the function takes both positive and negative values. We define a fixed order (homogeneous) derivative to be any combination of derivatives of fixed order n of the form

$$\phi^{(n)} = \sum_{|\alpha|=n} c_i \partial^\alpha \phi. \tag{5}$$

We will assume that our wavelet ϕ satisfies a genericity condition with regard to the zeroes of its derivatives, namely that for any two fixed order derivatives $\phi^{(n)}$ and $\phi^{(m)}$ with n, m any non-negative integers, the regular zeroes of $\phi^{(m)}$ are not contained in the zeroes of $\phi^{(n)}$, and that both have regular zeroes for all n, m. The existence of zeroes for all fixed order derivatives $\phi^{(n)}$ can be easily checked to be a necessary condition for Theorem 1 below to hold.

The main theorem on recovery of an image $f(x)$ from edge information provides sufficient conditions for the Marr conjecture, assuming edge information is obtained from any wavelet

$\phi(x) \in \mathcal{P}$. Here and below, unique recovery (determination) of a function f means unique up to mutiplicative constants.

Theorem 1. *Assuming the above genericity condition on a given $\phi(x) \in \mathcal{P}$ and given any sequence of scales $\sigma_i \xrightarrow[i\to\infty]{} \infty$, any function $f \in \mathcal{P}'$ is uniquely determined (up to constant multiples) by the multiscale edges of its wavelet transform, i.e., the zeroes of the family $\{f(x) * \phi(x/\sigma_i)\}_{i=1}^{\infty}$.*

This theorem is the strongest of its kind in two ways. First, it fails to hold if the exponential decay condition on f is weakened to algebraic decay - this still leaves the question open for the small class of functions such as $e^{-\ln^2 x}$ whose decay is between algebraic and exponential.

Theorem 2. *The above theorem fails to hold if the condition $f \in \mathcal{P}'$ is weakened to a requirement of algebraic decay, i.e., that $|f(x)| \leq C|x|^{-a}$ for some $a > 0$.*

Second, we can show that the above genericity condition on the zeroes of ϕ and its derivatives is the best of its kind, in that minimal weakenings of the condition make the theorem false. For example, if the genericity condition is weakened to require only that, for all fixed order derivatives $\phi^{(n)}$ and $\phi^{(m)}$, the regular zeroes of $\phi^{(n)}$ are not contained in the *regular* zeroes of $\phi^{(m)}$, then simple one-dimensional counterexamples exist.

As a simple corollary, the function f is of course uniquely determined by the zero set of its transform $f * \phi_\sigma(x)$ (i.e., assuming the zero set is known for all $\sigma > 0$) and also by its dyadic nodes, (i.e., in the special case where $\sigma_i = 2^i$ above). For the case of the one dimensional Ricker wavelet $\phi(x) = \frac{d^2}{dx^2}G$, where $G = (2\pi)^{-1/2)}e^{-x^2/2}$ (Gaussian edge information), the genericity condition (non-containment of regular zeroes of one derivative within the zeroes of another) can be proved using a result of Schur [21], stating that the even Hermite polynomials $H_{2n}(x)$ and normalized odd polynomials $H_{2n+1}(x)/x$ are irreducible (cannot be factored) over the rationals. From this the Marr conjecture for recovery of $f(x)$ from dyadic Gaussian edges (zeroes of $f(x)*\phi(2^{-i}x)$) follows in one dimension. The multidimensional Gaussian wavelet case ($\phi = \Delta G(x)$ in $\mathbb{R}^d$) similarly reduces to showing that the zeroes of $\phi(x) = \Delta e^{-x^2/2}$ and of its derivatives (in d dimensions) satisfy the genericity conditions. This in turn reduces to proving the same genericity conditions for zeroes of a family of multivariate Hermite polynomials (so-called Laplace-Hermite polynomials), obtained as the polynomial coefficients of the (multivariate) derivatives $\partial^\alpha \phi(x)$.

IV. Additional results

In addition to the above results for general wavelets, there are some additional specific results related to the recovery of one dimensional signals under Gaussian edges (i.e., nodes of convolutions with the Ricker wavelet).

For the specialization to the Ricker (Gaussian derivative) wavelet, we first consider the anti-intuitive nature of the results, which would indicate that asymptotically large features in the image (i.e., the edges at arbitrarily high scales σ_i) are what allow it to be recovered. In fact any infinite sequence of scales σ_i (i.e., not necessarily just large scales) will lead to unique recovery, as long as the sequence does not converge to 0, i.e., to microscopic scales.

Corollary 1. *For recovery of one dimensional signals $f(x)$, $x \in \mathbb{R}$ using Gaussian edges,*

(a) Under the above hypotheses, unique recovery is possible from edge information for any infinite sequence of positive scales σ_i, as long as σ_i does not converge to 0, i.e. it has a non-zero limit point.

(b) The statement of the main theorem is in general false if $\sigma_i \xrightarrow[i\to\infty]{} 0$; in fact there are images $f \in \mathcal{P}'$ for which Gaussian edge information at arbitrarily small scales $\sigma_i \xrightarrow[i\to\infty]{} 0$ is insufficient to recover f.

V. Proofs using moment expansions

The above results are based on the duality of two bases for $\mathbb{R}^n$, the monomials $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ (the basis for Taylor expansions) and the dual basis of derivatives $\delta^{(\alpha)} = \partial^\alpha \delta_0$ of the Dirac delta distribution δ_0, defined by $\delta_0(g(x)) = \langle \delta_0, g(x)\rangle = g(0)$ for smooth g. This is based on the fact that when all quantities are well-defined, $\langle \partial^\alpha \delta_0, f(x)\rangle = \langle \delta_0, (-1)^{|\alpha|}\partial^\alpha f\rangle$, so $\langle \partial^\alpha \delta_0, x^\beta\rangle = (-1)^{|\alpha|}\alpha!\delta_{\alpha\beta}$, where $\alpha! = \alpha_1! \dots \alpha_d!$ and $\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$. In the proper distributional space of analytic functions, a function $f \in \mathcal{P}'$ can be expressed as

$$f(x) = \sum_\alpha c_\alpha \delta^{(\alpha)} \tag{6}$$

with equality when

$$c_\alpha = \frac{(-1)^\alpha}{\alpha!}\langle f, x^\alpha\rangle. \tag{7}$$

Thus the coefficients c_α determining f are multiples of moments of f, given as $\langle f, x^\alpha\rangle = \int_{\mathbb{R}^d} f(x)x^\alpha dx$. Such singular expansions have been used implicitly in the multipole expansions of electrodynamics, and have also been applied to study other systems of PDE [6], [19], [4], [13], [7], [23]. A general study appears in [5].

After convolution of both sides with ϕ_σ, (6) now takes the form

$$\sigma^{-d} f * \phi(x/\sigma) \sim \sigma^{-d} \sum_\alpha c_\alpha (-\sigma)^{-|\alpha|}\phi^{(\alpha)}(x/\sigma), \tag{8}$$

which forms an asymptotic expansion [5]. Specifically, partial sums summing up to $|\alpha| = N$ are correct to order σ^{-N-d} as $\sigma \to \infty$.

This can be extended to show that the above asymptotic series is locally uniform in $w = \frac{x}{\sigma}$, and also to show that such expansions are also correct as asymptotic expansions for functions f with only finitely many moments, up to as many terms in (8) as can be defined by (7).

For the following theorem we define functions $f \in \mathcal{P}'$ to be of *negative exponential order* , since functions in this class satisfy $|f(x)| \leq Ce^{-k|x|}$.

Theorem 3. *Let the wavelet $\phi \in \mathcal{P}$, (thus ϕ grows subexponentially) and let $f \in \mathcal{P}'$ (so f is of negative exponential order). Then the asymptotic expansion (8) is valid uniformly on bounded subsets of $w = x/\sigma$, i.e., the normalized difference*

$$\left| f * \phi_\sigma(x) - \sum_{|\alpha| \le N} c_\alpha (-1)^{|\alpha|} (\sigma)^{-|\alpha|-d} \phi^{(\alpha)}(x/\sigma) \right| = o(\sigma^{-N-d}),$$

locally uniformly in $w = x/\sigma$ for any nonnegative integer N.

This means that on any compact set W of w, the above difference is smaller than $C\sigma^{-N-d}$ (as $\sigma \to \infty$) for arbitrarily small $C > 0$. This uniformity is needed to recover the initial condition f.

We renormalize $f * \phi_\sigma$ by a power of σ (which does not change the zeroes), defining (recall $x = \sigma w$)

$$F(\sigma, w) = \sigma^{n_0+d} (f * \phi_\sigma)(\sigma w).$$

Then by (8) as $\sigma \to \infty$,

$$F(\sigma, w) \sim \sum_{|\alpha| \ge n_0} (-1)^{|\alpha|} c_\alpha \sigma^{n_0 - |\alpha|} \phi^{(\alpha)}(w),$$

where n_0 is the lowest order $|\alpha|$ appearing in the above sum. Thus, locally uniformly in w, the series converges to its lowest term,

$$F(\sigma, w) \underset{\sigma \to \infty}{\longrightarrow} F(w) = (-1)^{n_0} \sum_{|\alpha| = n_0} \frac{c_\alpha}{\alpha!} \phi^{(\alpha)}(w). \tag{9}$$

Correspondingly, the zeroes of $F(\sigma, w)$ converge to those of $F(w)$. Note in particular that the zeroes of $F(\sigma, w)$ stabilize as $\sigma \to \infty$ in the variable $w = x/\sigma$. This means that the zero curves in the x variable itself move out to ∞; the normalization of x by dividing by σ is thus needed for the formation of stable limits of the node contours. We denote the node points w with $F(w) = 0$ the *asymptotic zero set* E.

Essentially it is the known information regarding convergence of the w-zero sets E_j of $F(\sigma_j, w)$ to their limit E (the zero set of F) together with their effective rates of convergence, that will uniquely determine the initial condition $f(x)$. Specifically, let $w' \in E$ be an asymptotic zero that is regular (i.e. transverse, so $F(w)$ takes both signs in all neighborhoods of w'). Then there is a sequence $w_j \in E_j$ with $w_j \to w'$. Using uniformity of the asymptotic expansion and replacing w by w_j, we obtain

$$0 = F(\sigma_j, w_j) \sim \sum_{|\alpha| \ge n_0} (-1)^{|\alpha|} c_\alpha \sigma^{n_0 - |\alpha|} \phi^{(\alpha)}(w_j).$$

This asymptotic statement means that the above expansion in σ converges to 0 as $\sigma \to \infty$ to all orders σ^{-k} for $k > 0$:

$$\lim_{j \to \infty} \sigma_j^l \sum_{n_0 \le |\alpha| \le n_0 + l} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma_j^{n_0 - |\alpha|} \phi^{(\alpha)}(w_j) = 0.$$

We now separate the sum on the left side above into the highest and lower powers σ_j^l (note that including the term σ_j^l in front, the total power of σ_j in this highest term is 0). Thus we have (note $w_j \underset{j \to \infty}{\to} w'$)

$$\sum_{|\alpha| = n_0 + l} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma^0 c_\alpha \phi^{(\alpha)}(w') + \lim_{j \to \infty} \sum_{n_0 \le |\alpha| < n_0 + l} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma^{n_0 + l - |\alpha|} \phi^{(\alpha)}(w_j) = 0.$$

This equation yields a recursion giving the moments c_α of f with order $|\alpha| = n_0 + l$ (in the first sum above), in terms of the lower order moments c_α in the second sum. Assuming as an induction step that all moments c_α for f are known for $|\alpha| < n_0 + k$, then this equation uniquely determines the sum

$$\sum_{|\alpha| = n_0 + l} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma^0 c_\alpha \phi^{(\alpha)}(w') \tag{10}$$

for all w' in the regular zero set of $F(w)$, and note that (10) is a homogeneous derivative of ϕ, i.e., a linear combination of derivatives of fixed order $n_0 + l$. Assuming (genericity condition) that the regular zero sets of the different homogeneous derivatives $\phi^n(w)$ of this type are never contained in the zero sets of any other homogeneous derivatives $\phi^m(w)$, it follows that the collection of $\phi^{(\alpha)}(w')$ (for $|\alpha| = n_0 + l$ fixed) are linearly independent on the asymptotic edge E (the zero set of F). Hence knowing the sum (10) on $w' \in E$ uniquely determines the coefficients (moments) c_α of order $|\alpha| = n_0 + l$, completing the inductive step of the recursion for determining the moments c_α of f.

Recovering f then becomes a moment problem for f, which is solvable for functions $f \in \mathcal{P}'$, which have negative exponential order.

On the other hand, if f fails to have exponential decay and instead has only algebraic decay (see above), it can be shown that the Marr conjecture (even for Gaussian derivative wavelets ϕ) does not hold.

VI. Geometry of heat equation nodes

The Marr conjecture for the Ricker (Gaussian derivative) wavelet has traditionally been studied using properties of heat equation solutions [16], [17], [3], [8], [9], [24]. Our results (see above) on unique recovery under the one dimensional Gaussian derivative wavelet involving sequences of scales σ_i that have positive limits or limit points, as well as the negative results for scales σ_i that converge to 0, also depend on the geometry of solutions to the heat equation.

These follow from some new results described below, related to nodes (zeroes) of solutions $u(x, t)$ of the one dimensional heat equation $\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$ for $x \in \mathbb{R}$, $t \ge 0$, with initial condition $u(x, 0) = f(x)$. The nodes of $\Delta u(x, t) = \frac{d^2}{dx^2} u(x, t)$ (which we denote as edge contours) are algebraic curves with strong analyticity properties. We mention some new results on such heat equation solutions, in particular that as t increases, new nodes cannot appear spontaneously. This extends prior results on this topic [1], [24], [2], [8], [11], [22].

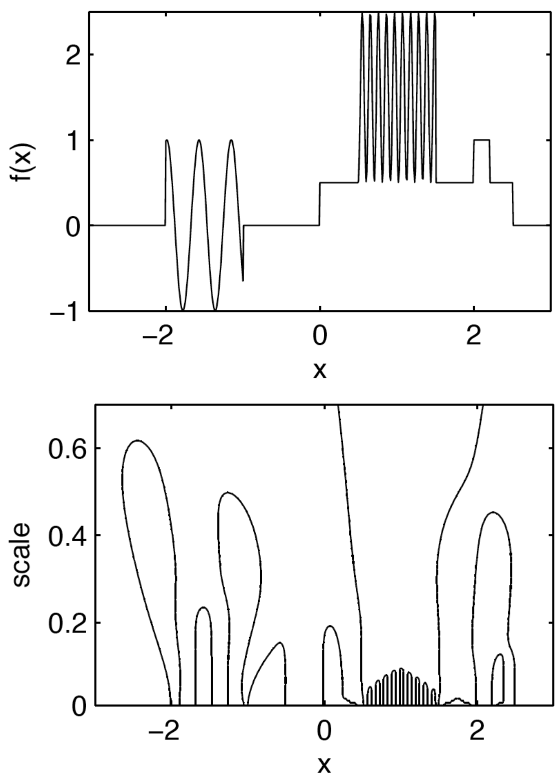

Fig. 2. A one dimensional image and its Gaussian edges at increasing scale σ (vertical axis).

Theorem 4. *If the initial condition f is an L^1 function of negative exponential order (i.e. $f(x) \leq Ce^{-k|x|}$ for some $C, k > 0$) then the edge contours of $u(x,t)$ at a fixed time t_2 form a subset of those at time $t_1 < t_2$.*

Note that if $u(x,t) = \Delta v(x,t)$ (with Δ now the one dimensional Laplacian) and v has an initial condition $g(x)$ with negative exponential order, then the edge contours of $v(x,t)$ are the nodes of $u(x,t)$. Since v can be recovered from these curves, so can u, giving:

Corollary 2. *If $u(x,t)$ is a solution of the one dimensional heat equation with initial condition f whose second integral $g(x) = \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz\ f(z)$ has negative exponential order, then $u(x,t)$ is uniquely determined by its nodes for $t > 0$, and more generally by its nodes at any infinite sequence of times $t_i > 0$ that does not converge to 0.*

VII. Conclusion

Though this paper reduces the general Marr conjecture to genericity conditions on the zeroes of the wavelet and its derivatives, it would be interesting to study these conditions in particular for the Ricker wavelet $\phi = \Delta G$ with G the Gaussian. As mentioned, this is equivalent to genericity of zeroes of the Laplace-Hermite polynomials, which we have proved in one dimension (in which the Laplace-Hermites are just Hermite polynomials of order 2 or more); proof of the genricity condition for the Laplace-Hermite polynomials in higher dimensions remains open.

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