Linear Algebra. By Peter Lax. John Wiley, 1997, xiv + 250 pp., $89.95. ISBN 0-471-11111-2.

Reviewed by **Mark A. Kon**

What is linear algebra? Perhaps it is most simply described as the study of finite-dimensional vector spaces and the matrices that act on them as linear operators. This subject seemed to have been conquered a few decades ago until the advent of high-powered computers and the detailed study of computational methods. Computers are suited to linear operations, and nonlinear operations can often be reduced to sequences of linear ones, for example, in solving nonlinear differential equations using Runge-Kutta and other algorithms. The demand in numerical mathematics for such things as solving systems of linear equations, finding eigenvalues, and inverting matrices has led to a torrent of research on analysis of algorithms that implement these operations efficiently. Research into these questions continues unabated today.

A curious fact about linear algebra is that for a long time, it has been both everywhere and nowhere in advanced mathematics. Though by implication linear algebra focuses on finite dimensional spaces, functional analysis has to an extent co-opted and appropriated much of the thought and language of linear algebra. The spectral theory done in functional analysis is more general and suited to normed linear spaces, and it contains much of finite dimensional spectral theory. Nevertheless, there is also much in finite dimensional linear algebra that is lost in its functional-analytic version, such as determinants, characteristic polynomials, and canonical forms.

In any case, linear algebra is a basic building block of mathematics. Mathematicians accept that we and others who use mathematics on any level above calculus should be thoroughly familiar with it. Moreover, it is "effective": if one runs across a problem that can be put in a linear algebra context, one will generally find the tools for tackling it.

The importance of linear algebra in analysis was made especially clear to me in an undergraduate-level applied mathematics course I taught recently. It focused on wavelet theory, a subject whose basic concepts are very usefully developed using the viewpoint of linear algebra. The notion of a multiresolution analysis in wavelet theory involves sequences of nested vector spaces, and many of the "pedestrian" terms of linear algebra come up in short order. Concepts involving linear independence, spanning, orthogonality, complementary spaces, and other topics in basic linear algebra were introduced in my class in very natural and practical contexts.

Linear algebra texts in recent years have been predominantly for undergraduates and have been dominated by two categories of text. The first is a "practical" yet generally rigorous approach geared to the second or third year student in engineering, computer science, or the physical sciences. The topics in such books have become very standardized; they include basic vector space theory (linear dependence, subspaces, etc.), linear systems, eigenvalues and eigenvectors, and perhaps things such as singular value decompositions and the concept of an abstract vector space. The philosophy behind such texts is that linear algebra is an important applied discipline that should nevertheless be presented in the context of a rigorous mathematical footing. (This is in contrast to the presentation of calculus, at least in many American texts, in which rigor decidedly takes a back seat to applications.) The second category of text is geared to post-calculus undergraduate mathematics majors, and sometimes graduate students in applied areas, and has a more theoretical approach. This category has traditionally included more rigorous algebraically oriented books such as Nomizu [**5**] and analytically oriented books such as Hoffman and Kunze [**1**], which is still in use in many courses oriented toward undergraduate mathematics majors and advanced students. There are also some more recent books such as Lancaster and Tismenetsky [**4**] and Horn and Johnson [**2**], [**3**] that address audiences at a more advanced level.

The book under review, largely based on a course the author has taught at New York University for a number of years, is aimed at graduate students who have already had a beginning linear algebra course. Its scope goes well beyond that of [**1**] and [**5**], and it is more analytically oriented than [**2**], [**3**], and [**4**]. The topics chosen are obviously the favorite ones of an experienced applied mathematician. One of its aims is to restore theory to its rightful place in applied linear algebra, a subject whose algorithmic aspects have come to dominate its teaching. A second aim is to present a rich base of applications, and a third is to describe some unusual numerical algorithms. This book reflects the author's view that the amount of known interesting mathematics is exploding, and that texts should maintain contact with the new and important aspects of mathematics. In this text Lax puts advanced material within reach of the beginning graduate student.

The beginning of Lax's book includes a rapid review of the basic theory. Notions involving linear independence, dimension, duality, and linear transformations are defined, and some basic relationships are derived, such as the standard relation between the dimensions of the kernel and the range. Lax relies on the technique of quotient spaces to avoid variable-counting in the discussion of linear systems and to make the proofs brief and transparent. These basic bones are fleshed out by highly nontrivial applications such as quadrature formulas, interpolation by polynomials, and the solution of the discrete Dirichlet problem. All this is in the first thirty pages or so, and the remainder of the book proceeds at a similar pace.

This book makes an effort to communicate some of the more unintuitive but nonetheless important concepts in linear algebra that illustrate the mystery of the dazzling connections between abstract mathematics and the real world. Two such concepts are determinants and the spectrum of a matrix. These fundamental ideas, which truly underlie so many aspects of nature, do not naturally exist at the most basic levels of human understanding. That is, they are examples of constructed concepts that become familiar and natural for those who work with them, but might seem unmotivated to those who first set eyes on them. Mathematicians think of determinants as second nature because of their widespread occurrence in fundamental mathematical constructions, but their definition is somewhat unintuitive. Similarly, spectral theory is, among other things, at the heart of the physics of quantum reality. It is a tribute to human ingenuity that something so unexpectedly important, whose definition might seem unmotivated at first sight, can now be a matter of common intuition for some. The author of a linear algebra book has to deal with the fact that, however intuitively clear such concepts may be to the expert, their motivation may seem tenuous to the uninitiated.

Lax's motivation for determinants is based on volumes of simplices, which to me is the most natural introduction to this subject. This geometric context has as an immediate algebraic consequence that the determinant is a multi-linear alternating function of its vector arguments. All other properties follow from this (together with the normalization $\det I = 1$), including of course the standard formula for the determinant.

Lax's motivation for spectral theory is quite appealing. He gives a deceptively innocuous example involving the behavior of high powers of matrices, motivated by the study of linear dynamical systems. It is worth repeating the example here: the four matrices are

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ -5 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 7 \\ -3 & -4 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 6.9 \\ -3 & -4 \end{pmatrix}.$$

Lax points out that

$$A^{1024} > 10^{700}, \quad B^{1024} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C^{1024} = \begin{pmatrix} -5 & -7 \\ 3 & 4 \end{pmatrix}, \quad D^{1024} < 10^{-78},$$

where the first inequality indicates that each term of the matrix is greater than 10^{700} in magnitude, and an analogous interpretation holds for the last one. The second and third cases are simply explained by observations that $B^2 = -I$ and $C^3 = -I$, but the key to the explanations of the first and fourth cases is the notion of eigenvectors and eigenvalues.

For me it was nice to see a very simple connection of spectral theory with "real life", if the latter can be assumed to include basic matrix maniuplations. There has been a strong tradition (especially in advanced mathematics) of exercising students' learning ability with presentation of seemingly unmotivated definitions. The result has been that the minority of students with decidedly abstract inclinations have been skilled at picking up the definitions, while students who benefit from real world connections with their new concepts have been largely left out. Lax's ability to present natural connections for unintuitive concepts is something that distinguishes this book.

There is also an interesting chapter on parameterized families of vectors and matrices. The main result is that if a matrix depends differentiably on a parameter, so do its simple eigenvalues. The subject of continuous matrix-valued functions has been important in applied mathematics, for example, in its infinite-dimensional version in the study of the anharmonic oscillator and its eigenvalue structure, but I did not expect to see this subject in a text on linear algebra.

A remarkable result with a simple explanation is the phenomenon of avoidance of crossing, where two parameterized eigenvalues seem headed for a crossing but, as in a game of chicken, swerve and avoid each other at the last minute. The explanation is a counting argument that shows that in the space of symmetric matrices, the matrices with multiple eigenvalues form a submanifold of codimension 2. Thus a one-parameter curve typically approaches but then moves away from this manifold without ever intersecting it.

The book is oriented toward operator theory, and aspects of matrix algebra that do not fit into this framework are largely omitted. For example, one finds no mention of the utility of looking at matrices as sums of outer products. On the other hand, some topics in the book can be viewed as nice introductions to their infinite-dimensional cousins as studied in functional analysis. For example, the Hahn-Banach theorem is presented as an example of convexity results, and Perron's theorem, which states that a positive matrix has a positive eigenvector corresponding to its largest eigenvalue, is an analog of the positivity of fundamental eigenfunctions of positive differential operators (after taking inverses). However, finite dimensional linear algebra is never far behind, for example, with Caratheodory's theorem, which states that every point of a convex set K in an n-dimensional space can be represented as a convex combination of $n + 1$ extreme points of K.

The proofs are almost always brief and to the point, and one might guess that the author enjoyed finding pleasing and simple proofs for standard results in linear algebra. The Gram-Schmidt orthogonalization method is derived in a seven-line proof using its definition as a recursive algorithm. The Jordan canonical form and completeness of eigenvectors and eigenvalues are proved using novel and striking techniques.

The applications often closely follow the theory. There is an elegant chapter on kinematics and dynamics involving matrix and vector equations. Rotation matrices and their infinitesimal generators, a basic example of Lie groups and algebras, are presented in the context of dynamics, and the introduction of time dependence leads to notions of instantaneous axes of rotation and angular velocities. Multi-particle vibrational systems are also studied in terms of matrix equations. Perron's theorem is applied to basic evolutionary biology models, in which limits of powers of stochastic matrices are shown to converge to positive multiples of the dominant eigenvector. Other sections deal with applications to economics and game theory.

Eight appendices, each only a few pages long, present surprising material not usually found in current texts. They include special determinants, symplectic matrices, tensor products, Gershgorin's theorem, and the multiplicity of eigenvalues. One appendix deals with Pfaff's theorem, which states that the determinant of an antisymmetric matrix of even order is the square of a homogeneous polynomial in its entries. Another appendix studies fast matrix multiplication and the possibility of multiplying $n \times n$ matrices in less than $O(n^3)$ multiplication steps. This is done through an analysis of Strassen's algorithm, which gives a power of $\log_2 7 = 2.807\ldots$ to replace 3 in the complexity order. Fast matrix multiplication has become something of a sport in numerical analysis and computer science, with numerous algorithms that have improved on Strassen's algorithm and on each other.

One interesting touch that I have not seen elsewhere in textbooks is the presence of headings on the upper right hand side of each page that reflect the current discussion on that page, regardless of section headings. It is quite eye-catching, though not surprising in these days of computer-generated text. It would also have been nice to include the chapter number on each page of the text.

Now some comments for the potential adopter of this book as a textbook for a class. This book is not for the uninitiated student. It assumes some prior experience with linear algebra, and it is written in a terse and informal style that is reminiscent of lecture notes rather than a polished text. There are some lapses in correct language; for example, the trivial subspace does not have any dimension according to the author's definition of dimension. The author uses the term "positive matrix" in two different ways (though he gives a warning regarding this), and he uses two definitions of direct sum of vector spaces (one on page 4 and one on page 7). The style is sometimes compressed; for example, the definition of null space includes the fact that it is a subspace, something which technically requires proof. Also, there are more than the usual number of typographical errors; students should be cautioned about these as well.

For these reasons, I recommend examination of the book before adoption. Nonetheless, the book is worthwhile as a text for students who are sophisticated enough to benefit from its rapid-fire style. Ambitious advanced undergraduates should benefit from this book as much as graduate students; indeed, some undergraduate programs have adopted this book for their talented students.

The exercises in this text are distributed throughout its body; among the 17 chapters of the book, the number of exercises per chapter ranges between two and ten, averaging around seven. In most cases this should provide a sufficient corpus of exercises for a typical graduate class, even if there is no bounty of additional exercises typical of an undergraduate text. Nevertheless, the instructor may need to add exercises to those in certain chapters.

A graduate text in linear algebra has been quite uncommon, at least in recent years; so there has been very little competition for this niche. This book proves that there are many topics to make such an effort worthwhile, and the text indeed fills a gap in the graduate curriculum. All in all, because of its directed and original approach and the overview it brings, the book is recommended for the teacher and researcher as well as for graduate students. In fact, I think that it has a place on every mathematician's bookshelf. The proofs are direct, novel, and elegant, and the presentation inspires one to rethink material that has sometimes become too routine.

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