Vectors in 2D and 3D

Vectors

1. Three dimensional coordinates:

Point in plane: two perpendicular lines as reference:
Vectors in 2D and 3D

Point in space: three perpendicular lines as reference -
Vectors in 2D and 3D

$x - y$ plane plus $z$ axis perpendicular to $x - y$ plane.

Coordinates of point $P : (x, y, z)$ indicated above [e.g., three corner lines of the room]

[we get the three numbers $(x, y, z)$ by dropping perpendiculars to the three axes]
Vectors in 2D and 3D

Ex 1: Calculation of an orbit to the moon:

[need to set up coordinate system and measure all points relative to this]
Vectors in 2D and 3D

[all mathematics is done by specifying position of the spacecraft and the moon relative to some coordinate system, say centered at the earth; that's how orbits are calculated at JPL and NASA]
Vectors in 2D and 3D

2. Vectors:

A vector is an arrow - it has direction and length. If you are hiking and say that you are 3 mi NNW of your camp you are specifying a vector.
Vectors in 2D and 3D

The precise mathematical statement is that:

**Geometric definition of vectors:** A vector is a directed line segment. The length of a vector \( \mathbf{v} \) is sometimes called its *magnitude* or the *norm* of \( \mathbf{v} \). We will always abbreviate length by the symbol

\[
\text{length of } \mathbf{v} = |\mathbf{v}|.
\]
Vectors in 2D and 3D

Two vectors are equal if they point in the same direction and have the same length:

[where the vector starts is not important]
Vectors in 2D and 3D
We can add vectors:

parallellogram rule

or
Vectors in 2D and 3D

And we can multiply vectors by real numbers (scalar multiplication):

If $\alpha > 0$, then $\alpha \mathbf{a}$ is the vector in the direction of $\mathbf{a}$ whose length is $\alpha \mathbf{a}$.

If $\alpha < 0$ then $\alpha \mathbf{a}$ satisfies

$$\text{direction } (\alpha \mathbf{a}) = - \text{ direction } (\mathbf{a}).$$

$$|\alpha \mathbf{a}| = |\alpha| |\mathbf{a}|.$$
Vectors in 2D and 3D
Since we can add vectors and perform scalar multiplication, we can subtract two vectors:

\[ \mathbf{a} - \mathbf{b} = \mathbf{a} + -\mathbf{b} \]

Now assume initial points of all vectors are located at the origin:
Vectors in 2D and 3D
Vectors in 2D and 3D
Consider also the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:
Vectors in 2D and 3D
Vectors in 2D and 3D

[We see \( a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \) and \( a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \)]

Notation: \( \mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \) = vector from origin to \((a_1, a_2, a_3)\).

Given the vector \( \mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \), we get by distance formula:

\[
|\mathbf{a}| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]
Vectors in 2D and 3D

Analytic definition of vectors in 3 dimensions: A vector is an array of numbers.

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\]
3. Addition of vectors

Analytic addition of vectors:

\[
\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]
Vectors in 2D and 3D

\[ \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} \]

Similarly in three dimensions:

fig 5
Vectors in 2D and 3D

\[
\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

\[
\Rightarrow \quad \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}
\]

Moreover, can show similarly that:

\[
\alpha \mathbf{a} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{bmatrix}
\]
Example 2:

A light plane flies at a heading of due north (direction which airplane is pointed) at air speed (speed relative to the air) of 120 km/hr in a wind blowing due east at 50 km/hr. What direction and speed does the plane move at relative to the ground?

A: Define the velocity of the airplane as the vector $\mathbf{v}$ whose length is the speed of the plane and whose direction is the direction of the plane:
Vectors in 2D and 3D

Note: wind velocity is $\begin{bmatrix} 50 \\ 0 \end{bmatrix}$; plane's air velocity (relative to wind) is $\begin{bmatrix} 0 \\ 120 \end{bmatrix}$

Also: every hour plane flies 120 kilometers north and 50 kilometers east.

Thus direction of the plane is same as $\begin{bmatrix} 50 \\ 120 \end{bmatrix}$
Vectors in 2D and 3D

What is the speed: \[ \sqrt{120^2 + 50^2} = 130. \]
Vectors in 2D and 3D

Thus speed of airplane is $130 = \text{length of } \begin{bmatrix} 50 \\ 120 \end{bmatrix}$; 

direction of airplane is same as $\begin{bmatrix} 50 \\ 120 \end{bmatrix}$

$\Rightarrow$ velocity of airplane is vector

$\begin{bmatrix} 50 \\ 120 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 120 \end{bmatrix} = \text{wind velocity} + \text{air velocity of plane}$
Vectors in 2D and 3D

**Moral:** when we take an object moving at velocity \( \mathbf{v}_1 \) (air velocity) in a medium which is moving at velocity \( \mathbf{v}_2 \) (wind velocity), the total velocity of the object is \( \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \).
Properties of Vectors

4. Properties of vectors:
Theorem: If \( a, b, \) and \( c \) are vectors (in two or three dimensions) and \( d \) and \( e \) are scalars, then:
(a) \( a + b = b + a \) (commutativity)
(b) \( (a + b) + c = a + (b + c) \) (associativity)
(c) \( d(a + b) = da + db \) (distributivity)
(d) \( (d + e)a = da + ea \) (distributivity)
(e) There is a unique vector \( 0 \) such that \( a + 0 = a \) for all vectors \( a \)
(f) For every vector \( a \) there is a unique vector \( -a \) such that \( a + -a = 0 \)
(g) \( d(ea) = (de)a \)
(h) \( 1 \cdot a = a \)
Properties of Vectors

[other properties you would expect are listed in the book; they follow from the above properties]

Proof: (a): Assume that

\[
a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}; \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}
\]

Note that the entries in \(a\) have the general form \(a_i\) for \(i = 1\) through \(n\).
Properties of Vectors

We will use shorthand by writing $\mathbf{a}$ in terms of the general term in $\mathbf{a}$:

$$\mathbf{a} = (a_i).$$

Similarly,

$$\mathbf{b} = (b_i).$$

Thus

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ \vdots \\ b_n + a_n \end{bmatrix} = \mathbf{b} + \mathbf{a}.$$
Properties of Vectors

In shorthand the above could be written:

$$a + b = (a_i) + (b_i) = (a_i + b_i) = (b_i + a_i) = (b_i) + (a_i) = b + a.$$ 

(b) 

$$\begin{align*}
(a + b) + c &= (((a_i) + (b_i)) + (c_i)) = (a_i + b_i) + (c_i) \\
&= ((a_i + b_i) + c_i) = (a_i + (b_i + c_i)) = (a_i) + (b_i + c_i) \\
&= a_i + ((b_i) + (c_i)) = a + (b + c) \text{ as desired.}
\end{align*}$$
Properties of Vectors

As an exercise, write out the proof in full, i.e. replace

\[
\begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_n \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    a_1 + b_1 \\
    a_2 + b_2 \\
    \vdots \\
    a_n + b_n \\
\end{bmatrix}
\]

e tc., to see how it looks.
Properties of Vectors

[Note it is remarkable that if you replace the word vector with the word matrix, the same statements as above are all still true.]
Vectors in n dimensions

Analytic definition of vectors in $n$ dimensions: A vector is a vertical array of $n$ numbers:

$$
v = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}.
$$

Note: all definitions of analytic operations on vectors in 3 dimensions hold for vectors in $n$ dimensions.
Vectors in $n$ dimensions

Can easily see that all properties (1)-(8) of vectors in 3 and 2 dimensions carry over to vectors in $n$ dimensions; proofs are identical.

[geometric visualization of vectors in $n$ dimensions is not necessary at this point]
Vector spans

5. Spans of vectors

Def 6: We define $\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_i \in \mathbb{R} \right\}$

where $\mathbb{R}$ means the set of all real numbers. Thus $\mathbb{R}^3$ is all 3-tuples of real numbers.

Def 7: A *linear combination* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a sum $c_1 \mathbf{a} + c_2 \mathbf{b}$

for constants $c_1$ and $c_2$.

Linear combination for larger collection of vectors works the same way.
Vector spans

Ex: Consider two vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1$; $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2$. 
Vector spans

Then:

\[
C_1 a + C_2 b = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ 0 \end{bmatrix}.
\]

Note as \(C_1\) and \(C_2\) vary, this covers all points in the \(x - y\) plane.

We say \textit{span} of \(a\) and \(b\) \(= all possible linear combinations of \(a\) and \(b\) \(= all vectors in \(x - y\) plane.\)
Vector spans

Generally can show: span of two vectors = all vectors contained in the plane of the first two.

**General definition:** The *span* of a collection of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \) is the collection of all possible linear combinations

\[
\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \left\{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n : c_i \in \mathbb{R}\right\}.
\]

Above \( \in \) means "is an element of".
Matrices and linear systems

6. Matrices and linear systems of equations:
Return to previous problem: Coal, Steel, Electricity:

[NOTE: There is an example of Leontief's economic model in the book; the equations there are DIFFERENT than here; they guarantee that the amount used for the production of each commodity (coal, steel, electricity) ends up equalling the value of the commodity produced; in our variation of Leontief's equations we have (non-zero) targets for net production of each commodity, and find production levels to attain those targets]
Matrices and linear systems

To make $1 of coal, takes no coal, $.10 of steel, $.10 of electricity.
To make $1 of steel, it takes $.20 of coal, $.10 of steel, and $.20 of electricity.
To make $1 of electricity, it takes $.40 of coal; $.20 of steel, and $.10 of electricity.

Let

\[ x_1 = \text{total amount of coal produced} \]
\[ x_2 = \text{total amount of steel produced} \]
\[ x_3 = \text{total amount of electricity produced} \]
Matrices and linear systems

Goals of economy:

If we want the economy to output $1 billion of coal, $.7 billion of steel, and $2.9 billion of electricity, how much coal, steel, and electricity will we need to use up? I.E., what will \( x_1, x_2, x_3 \) be?

Recall the equations:

\[
\begin{align*}
x_1 - .2x_2 - .4x_3 &= 1 \\
- .1x_1 + .9x_2 - .2x_3 &= .7 \\
- .1x_1 - .2x_2 + .9x_3 &= 2.9
\end{align*}
\]
Matrices and linear systems

give us $x_i$ for $i = 1, 2, \text{ and } 3$.

Note: system (*) equivalent to a single matrix equation:

$$
\begin{bmatrix}
1 & -0.2 & -0.4 \\
-0.1 & 0.9 & -0.2 \\
-0.1 & -0.2 & 0.9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
0.7 \\
2.9
\end{bmatrix}
$$
Matrices and linear systems

Check:

\[
\begin{bmatrix}
  x_1 & -0.2x_2 & -0.4x_3 \\
  -0.1x_1 & +0.9x_2 & -0.2x_3 \\
  -0.1x_1 & -0.2x_2 & +0.9x_3
\end{bmatrix}
= \begin{bmatrix} 1 \\ 0.7 \\ 2.9 \end{bmatrix}
\]

\[
\Rightarrow \quad x_1 - 0.2x_2 - 0.4x_3 = 1
\]

\[
(*) \quad -0.1x_1 + 0.9x_2 - 0.2x_3 = 0.7,
\]

\[
-0.1x_1 - 0.2x_2 + 0.9x_3 = 2.9
\]
Matrices and linear systems

= original system of equations.

[Very compact form for writing equations.]

If we call

$$A = \begin{bmatrix}
1 & - .2 & - .4 \\
- .1 & .9 & - .2 \\
- .1 & - .2 & .9
\end{bmatrix},$$
Matrices and linear systems

\[
\mathbf{x} = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix}
  1 \\
  .7 \\
  2.9
\end{bmatrix},
\]

then the equation reads:

\[
A\mathbf{x} = \mathbf{b}.
\]

Thus can concisely describe systems using matrices.
Matrices and linear systems

Generally: if we have

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\
    &\vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]
Matrices and linear systems
Can be described by $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix} = \begin{bmatrix} a_{i,j} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$
Matrices and linear systems

[now we can concisely write any system of equations].
Matrix multiplication and vectors

7. Matrix multiplication and vectors.

Ex 1: System from above:

\[
\begin{bmatrix}
1 & -0.2 & -0.4 \\
-0.1 & 0.9 & -0.2 \\
-0.1 & -0.2 & 0.9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
0.7 \\
2.9
\end{bmatrix}
\]

\[Ax = b.\]

General form:
Matrix multiplication and vectors

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
=
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\]

Write:

\[A = [a_1 \ a_2 \ a_3]\]

where

\[
\begin{align*}
  a_1 &= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}; &
  a_2 &= \begin{bmatrix} -2 \\ .9 \\ -2 \end{bmatrix}; &
  a_3 &= \begin{bmatrix} -4 \\ -2 \\ .9 \end{bmatrix}.
\end{align*}
\]
Matrix multiplication and vectors

Focus on left side:

\[
A \mathbf{x} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
= \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]
Matrix multiplication and vectors

\[
\begin{bmatrix}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11}x_1 \\
  a_{21}x_1 \\
  a_{31}x_1 
\end{bmatrix}
+ \begin{bmatrix}
  a_{12}x_2 \\
  a_{22}x_2 \\
  a_{32}x_2 
\end{bmatrix}
+ \begin{bmatrix}
  a_{13}x_3 \\
  a_{23}x_3 \\
  a_{33}x_3 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} \\
  a_{21} \\
  a_{31} 
\end{bmatrix}
+ \begin{bmatrix}
  a_{12} \\
  a_{22} \\
  a_{32} 
\end{bmatrix}
+ \begin{bmatrix}
  a_{13} \\
  a_{23} \\
  a_{33} 
\end{bmatrix}
\]

\[
x_1 \begin{bmatrix}
  a_{11} \\
  a_{21} \\
  a_{31} 
\end{bmatrix}
+ x_2 \begin{bmatrix}
  a_{12} \\
  a_{22} \\
  a_{32} 
\end{bmatrix}
+ x_3 \begin{bmatrix}
  a_{13} \\
  a_{23} \\
  a_{33} 
\end{bmatrix}
\]
Matrix multiplication and vectors

\[ = x_1a_1 + x_2a_2 + x_3a_3 \]

Conclusion:

\[
Ax = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1a_1 + x_2a_2 + x_3a_3
\]

Ex 2:

\[
A = \begin{bmatrix}
1 & -0.2 & -0.4 \\
-0.1 & 0.9 & -0.2 \\
-0.1 & -0.2 & 0.9
\end{bmatrix}.
\]
Matrix multiplication and vectors

Then

\[ A \mathbf{x} = \begin{bmatrix}
1 & -0.2 & -0.4 \\
-0.1 & 0.9 & -0.2 \\
-0.1 & -0.2 & 0.9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} \]

\[ = x_1 \begin{bmatrix} 1 \\ -0.1 \\ -0.1 \end{bmatrix} + x_2 \begin{bmatrix} -0.2 \\ 0.9 \\ -0.2 \end{bmatrix} + x_3 \begin{bmatrix} -0.4 \\ -0.2 \\ 0.9 \end{bmatrix}. \]

[another way of writing it]
Matrix multiplication and vectors

Note that $Ax = b$ is:

\[
x_1 \begin{bmatrix} 1 \\ -1 \\ .1 \end{bmatrix} + x_2 \begin{bmatrix} -.2 \\ .9 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -.4 \\ -2 \\ .9 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \\ 2.9 \end{bmatrix}.
\]
8. Spanning and equations:

Ex 3: Consider question: What is the span of the vectors

\[ a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \text{ and } a_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \]?

Answer: Find all vectors \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) obtained as:

\[ c_1 a_1 + c_2 a_2 + c_3 a_3 = b \] (1)

for some \( c_1, c_2, c_3 \).
Spanning and equations

Equations read:

\[
\begin{bmatrix}
a_1 & a_2 & a_3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

or

\[A c = b \quad \text{where} \quad A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\
c_2 \\
c_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\
b_2 \\
b_3 \end{bmatrix}\]
Spanning and equations

\[
\begin{bmatrix}
1 & 2 & 0 \\
-1 & 0 & -2 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}.
\]

Question: for what \( b_1, b_2, b_3 \) is there a solution \( c_1, c_2, c_3 \) above.

[ System of 3 eqn's in 3 unknowns. ]
Spanning and equations

Augmented matrix:

\[
\begin{bmatrix}
  1 & 2 & 0 & | & b_1 \\
  -1 & 0 & -2 & | & b_2 \\
  0 & 1 & -1 & | & b_3 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
  0 & 2 & -2 & | & b_1 + b_2 \\
  0 & 1 & -1 & | & b_3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 2 & 0 & | & b_1 \\
  0 & 1 & -1 & | & b_1/2 + b_2/2 \\
  0 & 1 & -1 & | & b_3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 2 & 0 & | & b_1 \\
  0 & 1 & -1 & | & b_1/2 + b_2/2 \\
  0 & 1 & -1 & | & b_3 \\
\end{bmatrix}
\]
Spanning and equations

\[
\begin{bmatrix}
1 & 2 & 0 & b_1 \\
0 & 1 & -1 & b_1/2 + b_2/2 \\
0 & 0 & 0 & b_3 - b_1/2 - b_2/2
\end{bmatrix}
\]

(2)

Note: exists solution iff \( b_3 - b_1/2 - b_2/2 = 0 \).
Conclude solution does NOT exist for all \( b \).

Note that:
equation (1) has solution for ALL \( b \)
\[\iff\ every\ vector\ \( b \in \mathbb{R}^3 \) is a linear combination of \( a_1, a_2, a_3 \)
\[\iff\ a_1, a_2, a_3 \text{ span } \mathbb{R}^3.\]
Spanning and equations

[Note also: if left side of (2) had a pivot entry on last row, we wouldn't have problem with no solutions].

Conclude: also:

equation (1) has a solution for ALL \( b \) if the echelon form of \( A \) has a pivot entry in every row.

**Theorem 4:** The following are equivalent:

(a) The columns \( a_1, a_2, a_3 \) of \( A \) span \( \mathbb{R}^3 \)

(b) \( Ax = b \) has a solution for each \( b \in \mathbb{R}^3 \)

(c) \( A \) has a pivot position in every row.
Spanning and equations

Proof: In book.

More generally, this theorem holds for any set of vectors:

**Theorem 5:** The following are equivalent for a $m \times n$ matrix $A$:

(a) The columns $a_1, a_2, \ldots, a_n$ of a matrix $A$ span $\mathbb{R}^m$

(b) $Ax = b$ has a solution for each $b \in \mathbb{R}^n$

(c) $A$ has a pivot position in every row.
Spanning and equations

Note this gives an algorithm for checking whether vectors $a_1, \ldots, a_n$ span $\mathbb{R}^n$ - form matrix $A$ and check if every row has a pivot position.

[we now start material from section 1.5]
Properties of matrix-vector products:

**Theorem 6:** If \( A \) is a matrix and \( u \) and \( v \) are vectors and \( c \) is a scalar, then:

(a) \( A(u + v) = Au + Av \)

(b) \( A(cu) = cA(u) \).

**Proof:** (a): 

\[
A(u+v) = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots \\ u_n & v_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = Au + Av
\]
\begin{align*}
= (u_1 + v_1)a_1 + (u_2 + v_2)a_2 + \ldots + (u_n + v_n)a_n \\
= (u_1a_1 + \ldots + u_n a_n) + (v_1a_1 + \ldots + v_n a_n) \\
= Au + Av.
\end{align*}

(b) In book - proof is similar.