Homogeneous equations, Linear independence

1. Homogeneous equations:

Ex 1: Consider system:

$$\begin{aligned} x_1 \quad + 2x_2 \qquad\qquad &= 0 \\ -x_1 \qquad\qquad - 2x_3 &= 0 \\ x_2 \quad - x_3 &= 0 \end{aligned}$$

Matrix equation:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{3}$$

Homogeneous equation:

$$A\mathbf{x} = \mathbf{0}.$$

At least one solution:

$$\mathbf{x} = \mathbf{0}.$$

Other solutions called *nontrivial* solutions.

Theorem 1: *A nontrivial solution of* (3) *exists iff [if and only if] the system has at least one free variable in row echelon form. The same is true for any homogeneous system of equations.*

Proof: If there are no free variables, there is only one solution and that must be the trivial solution. Conversely, if there are free variables, then they can be non-zero, and there is a nontrivial solution. □

Ex 2: Reduce the system above:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{as before}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$

$$x_1 + 2x_2 = 0; \quad x_2 - x_3 = 0; \quad 0 = 0.$$

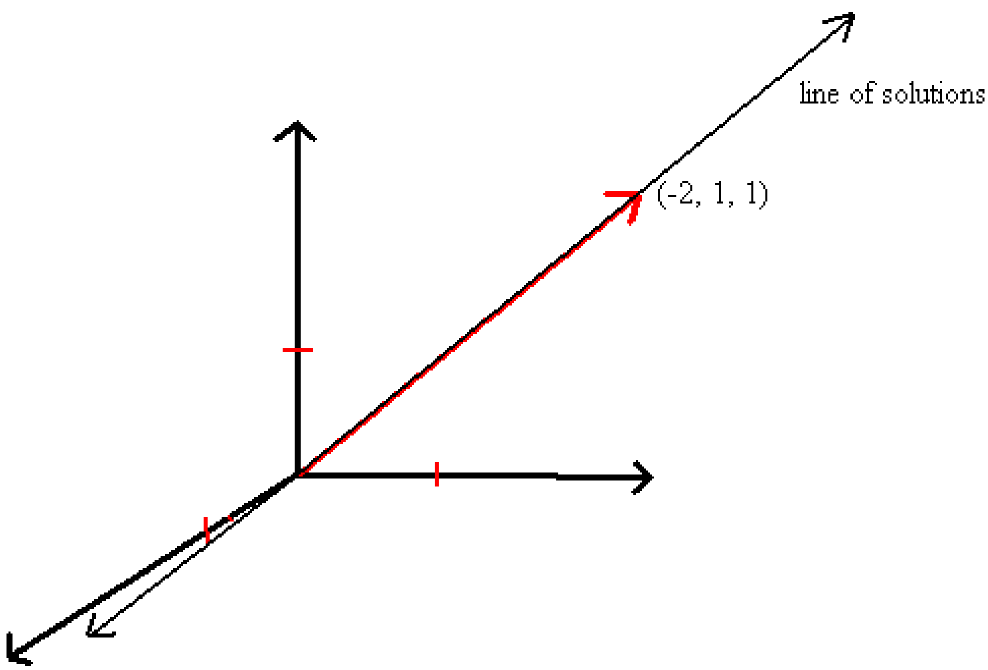
Note that $x_3 =$ free variable (non-pivot); hence general solution is

$$x_2 = x_3; \qquad x_1 = -2x_2 = -2x_3.$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

Parametric vector form of solution.

x_3 arbitrary: straight line -



Theorem 2: *A homogeneous system always has a nontrivial solution if the number of equations is less than the number of unknowns*.

Pf: If we perform a Gaussian elimination on the system, then the reduced augmented matrix has the form:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \dots & & & | & 0 \\ 0 & 0 & 1 & a_{24} & \dots & & | & 0 \\ 0 & 0 & 0 & 1 & a_{35} & \dots & | & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & | & \vdots \\ & & \vdots & & & & | & 0 \end{bmatrix}$$

with the remaining rows zeroes on the left side. If the number of equations is less than the number of unknowns, then not every column can have a 1 in it, so there are free variables. By previous theorem, there are nontrivial solutions. □

1. Inhomogeneous equations:

[we should briefly mention the relationship between homogeneous and inhomogeneous equations:]

Consider general system:

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

Suppose $\mathbf{p}$ is a particular solution of (1), so $A\mathbf{p} = \mathbf{b}$. If $\mathbf{x}$ is any other solution of (1), we still have $A\mathbf{x} = \mathbf{b}$. Subtracting the two equations:

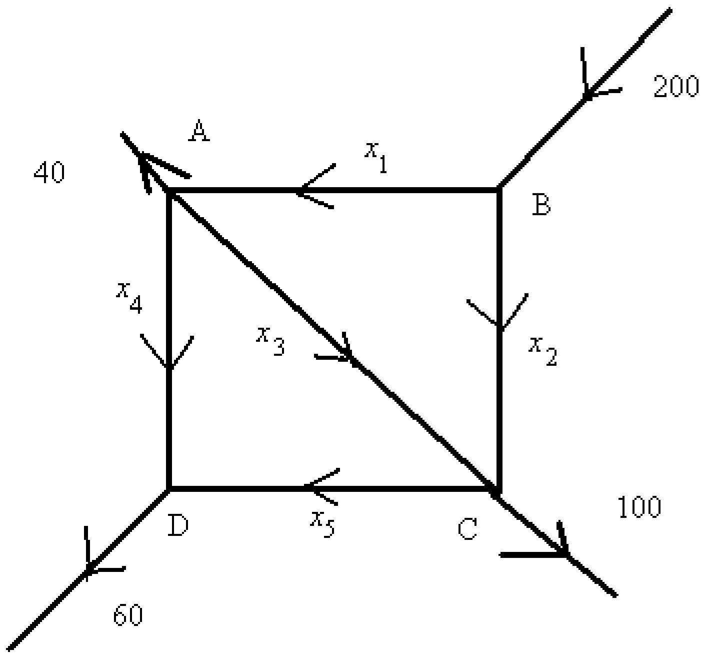
$$A\mathbf{x} - A\mathbf{p} = \mathbf{0} \qquad \Rightarrow \qquad A(\mathbf{x} - \mathbf{p}) = \mathbf{0}.$$

So $\mathbf{v}_h = \mathbf{x} - \mathbf{p}$ satisfies the homogeneous equation. Generally:

Theorem 1: *If $\mathbf{p}$ is a particular solution of (1), then for any other solution $\mathbf{x}$, we have that $\mathbf{v}_h = \mathbf{x} - \mathbf{p}$ solves the homogeneous equation (i.e., with $\mathbf{b} = \mathbf{0}$). Thus every solution $\mathbf{x}$ of (1) can be written $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$,* where *$\mathbf{v}_h$ is a solution of the homogeneous equation.*

2. Application: Network flows

Traffic pattern at Drummond Square:



Quantities in cars/min. What are the flows on the inside streets? One equation for each node:

$$x_1 - x_3 - x_4 = 40$$

$$-x_1 - x_2 = -200$$

$$x_2 + x_3 - x_5 = 100$$

$$x_4 + x_5 = 60$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & | & 40 \\ -1 & -1 & 0 & 0 & 0 & | & -200 \\ 0 & 1 & 1 & 0 & -1 & | & 100 \\ 0 & 0 & 0 & 1 & 1 & | & 60 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & -1 & -1 & -1 & 0 & -160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So:

$$x_1 = 100 + x_3 - x_5$$

$$x_2 = 100 - x_3 + x_5$$

$$x_4 = 60 - x_5,$$

where x_3, x_5 are free.

Constraint: if for example all flows have to be positive; then we require $x_i \geq 0$ for all i. Therefore:

$$x_3, x_5 \geq 0$$

$$-100 \leq x_3 - x_5 \leq 100$$

$$x_5 \leq 60$$

This corresponds to a region in the x_3, x_5 plane - can be plotted if desired.

If they closed off road x_3 and x_5, then we have $x_3 = x_5 = 0$, so that

$$x_1 = 100,\ x_2 = 100,\ x_4 = 60$$

note that then traffic flow becomes uniquely determined.

Definition 1:

A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is *linearly independent* if no vector in the collection is a linear combination of the others.

Equivalently,

Definition 2: A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is *linearly independent* if the only way we can have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$ is if all of the $c_i = 0$.

Equivalence of the definitions:

Def 1 $\Rightarrow$ Def 2

If no vector is a linear combination of the others, then if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

we will show that $c_1, \ldots, c_n$ have also to be $\mathbf{0}$.

Proof: Suppose not (for contradiction). Without loss of generality, assume $c_1 \neq 0$ (proof works same way otherwise). Then we have:

$$\mathbf{v}_1 = -c_2/c_1\ \mathbf{v}_2 - \ldots - c_n/c_1\ \mathbf{v}_n\ ,$$

contradicting that no vector is a combination of the others. Thus the c_i all have to be 0 as desired. □

Note: If S_2 is a collection of vectors and S_1 is a subcollection of S_2, then

If S_2 is linearly independent

$\Rightarrow$ no vector in S_2 is a linear combination of the others

$\Rightarrow$ no vector in S_1 is a linear combination of the others (since every vector in S_1 is also in S_2)

$\Rightarrow$ S_1 is linearly independent.

Logically equivalent [contrapositive]

If S_1 is linearly dependent (i.e., not independent)

$\Rightarrow$ S_2 is linearly dependent

[These are stated more formally in the book as theorems.]

Theorem 2: *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a collection of vectors in $\mathbb{R}^d$. Then S is linearly dependent if and only if one of the vectors $\mathbf{v}_i$ is a linear combination of the previous ones $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$.*

Proof: ($\Rightarrow$) If S is linearly dependent, then there is a set of constnats c_i *not* all 0 such that

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}.$$

Let c_k be the last non-zero coefficient. Then the rest of the coefficients are zero, and

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k = \mathbf{0}$$

$$\rightarrow \quad \mathbf{v}_k = -c_1/c_k\,\mathbf{v}_1 - c_2/c_k\,\mathbf{v}_2 - \ldots - c_{k-1}/c_k\,\mathbf{v}_{k-1}$$

i.e. one of the vectors is a linear combination of the previous ones.
($\Leftarrow$) Obvious.

3. Checking for linear independence:

Example 2: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Are they linearly independent?

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$\Rightarrow$

$$c_1 + c_2 - c_3 = 0$$

$$c_1 - c_2 \quad = 0$$

Reduced matrix:

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 1 & -1 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 1 & | & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & -1/2 & | & 0 \end{bmatrix}$$

Conclude: there are free variables. By theorem on homogeneous equations there is a nontrivial solution, so c_i need not be 0.

Thus not all c_i must be 0 $\Rightarrow$ not linearly independent.

[note that if number of vectors is greater than the size of the vectors, this will always happen].

More generally thus:

Theorem 3: *In $\mathbb{R}^n$, if we have more than n vectors, they cannot be linearly independent.*

From above we have:

Algorithm: *To check whether vectors are linearly independent, form a matrix with them as columns, and row reduce.*

(a) If reduced matrix has free variables (i.e., $\exists$ a non-pivot column), then they are not independent.

(b) If there are no free variables (i.e., there are no nonpivot columns), they are independent.