1. Inhomogeneous equations:

[we should briefly mention the relationship between homogeneous and inhomogeneous equations:]

Consider general system:

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

Suppose **p** is a particular solution of (1), so $A\mathbf{p} = \mathbf{b}$. If **x** is any other solution of (1), we still have $A\mathbf{x} = \mathbf{b}$. Subtracting the two equations:

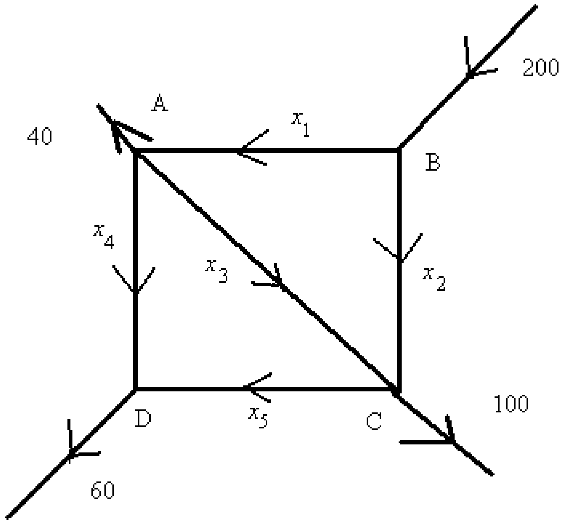
$$A\mathbf{x} - A\mathbf{p} = \mathbf{0} \qquad \Rightarrow \qquad A(\mathbf{x} - \mathbf{p}) = \mathbf{0}.$$

So $\mathbf{v}_h = \mathbf{x} - \mathbf{p}$ satisfies the homogeneous equation. Generally:

Theorem 1: *If* **p** *is a particular solution of* (1)*, then for any other solution* **x***, we have that* $\mathbf{v}_h = \mathbf{x} - \mathbf{p}$ *solves the homogeneous equation (i.e., with* $\mathbf{b} = \mathbf{0}$*). Thus every solution* **x** *of* (1) *can be written* $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ *is a solution of the homogeneous equation.*

2. Application: Network flows

Traffic pattern at Drummond Square:



Quantities in cars/min. What are the flows on the inside streets? One equation for each node:

$$x_1 - x_3 - x_4 = 40$$

$$-x_1 - x_2 = -200$$

$$x_2 + x_3 - x_5 = 100$$

$$x_4 + x_5 = 60$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & | & 40 \\ -1 & -1 & 0 & 0 & 0 & | & -200 \\ 0 & 1 & 1 & 0 & -1 & | & 100 \\ 0 & 0 & 0 & 1 & 1 & | & 60 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & -1 & -1 & -1 & 0 & -160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So:

$$x_1 = 100 + x_3 - x_5$$

$$x_2 = 100 - x_3 + x_5$$

$$x_4 = 60 - x_5,$$

where $x_3,\ x_5$ are free.

Constraint: if for example all flows have to be positive; then we require $x_i \geq 0$ for all i. Therefore:

$$x_3, x_5 \geq 0$$

$$-100 \leq x_3 - x_5 \leq 100$$

$$x_5 \leq 60$$

This corresponds to a region in the x_3, x_5 plane - can be plotted if desired.

If they closed off road x_3 and x_5, then we have $x_3 = x_5 = 0$, so that

$$x_1 = 100,\ x_2 = 100,\ x_4 = 60$$

note that then traffic flow becomes uniquely determined.

Definition 1:

A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is *linearly independent* if no vector in the collection is a linear combination of the others.

Equivalently,

Definition 2: A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is *linearly independent* if the only way we can have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$ is if all of the $c_i = 0$.

Equivalence of the definitions:

Def 1 $\Rightarrow$ Def 2

If no vector is a linear combination of the others, then if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

we will show that $c_1, \ldots, c_n$ have also to be **0**.

Proof: Suppose not (for contradiction). Without loss of generality, assume $c_1 \neq 0$ (proof works same way otherwise). Then we have:

$$\mathbf{v}_1 = -c_2/c_1\, \mathbf{v}_2 - \ldots - c_n/c_1\, \mathbf{v}_n\,,$$

contradicting that no vector is a combination of the others. Thus the c_i all have to be 0 as desired. □

Note: If S_2 is a collection of vectors and S_1 is a subcollection of S_2, then

If S_2 is linearly independent

$\Rightarrow$ no vector in S_2 is a linear combination of the others

$\Rightarrow$ no vector in S_1 is a linear combination of the others (since every vector in S_1 is also in S_2)

$\Rightarrow$ S_1 is linearly independent.

Logically equivalent [contrapositive]

If S_1 is linearly dependent (i.e., not independent)

$\Rightarrow$ S_2 is linearly dependent

[These are stated more formally in the book as theorems.]

Theorem 2: *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a collection of vectors in $\mathbb{R}^d$. Then S is linearly dependent if and only if one of the vectors $\mathbf{v}_i$ is a linear combination of the previous ones $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$.*

Proof: ($\Rightarrow$) If S is linearly dependent, then there is a set of constnats c_i *not* all 0 such that

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}.$$

Let c_k be the last non-zero coefficient. Then the rest of the coefficients are zero, and

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k = \mathbf{0}$$

$$\rightarrow \quad \mathbf{v}_k = -c_1/c_k\,\mathbf{v}_1 - c_2/c_k\,\mathbf{v}_2 - \ldots - c_{k-1}/c_k\,\mathbf{v}_{k-1}$$

i.e. one of the vectors is a linear combination of the previous ones.
($\Leftarrow$) Obvious.

3. Checking for linear independence:

Example 2: Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Are they linearly independent?

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$\Rightarrow$

$$c_1 + c_2 - c_3 = 0$$

$$c_1 - c_2 \quad\quad = 0$$

Reduced matrix:

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 1 & -1 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 1 & | & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & -1/2 & | & 0 \end{bmatrix}$$

Conclude: there are free variables. By theorem on homogeneous equations there is a nontrivial solution, so c_i need not be 0.

Thus not all c_i must be 0 $\Rightarrow$ not linearly independent.

[note that if number of vectors is greater than the size of the vectors, this will always happen].

More generally thus:

Theorem 3: *In $\mathbb{R}^n$, if we have more than n vectors, they cannot be linearly independent.*

From above we have:

Algorithm: *To check whether vectors are linearly independent, form a matrix with them as columns, and row reduce.*

(a) If reduced matrix has free variables (i.e., $\exists$ a non-pivot column), then they are not independent.

(b) If there are no free variables (i.e., there are no nonpivot columns), they are independent.

[material from section 1.8 starts here]

Lecture 4

1. A note on linear transformations: Single variable

Simple idea: $y = ax = T(x)$, where $x, y \in \mathbb{R}^1$

[e.g., $y =$ distance travelled by light, $x =$ time, $a =$ speed of light]

This is a linear function.

Properties: $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

$T(c\mathbf{x}) = cT(\mathbf{x})$

2. Note on linear functions: more variables

NOTE: Linear transformations are a simple and often accurate way of describing the transition from input to output. Here is an example.

In a chemical reaction three chemicals C_1, C_2, C_3 are mixed in weights x_1, x_2, x_3 grams. After one minute the amounts of the chemicals are

y_1, y_2, y_3, where

$$\begin{aligned} y_1 &= .5x_1 + .25x_2 + .25x_3 \\ y_2 &= .25x_1 + .25x_2 + .5x_3 \\ y_3 &= .25x_1 + .5x_2 + .25x_3 \end{aligned}$$

Notice that variables **y** are a linear function of variables **x**

We write $\mathbf{y} = T(\mathbf{x})$ if above true.

Of course we have matrix A such that $\mathbf{y} = A\mathbf{x} = T(\mathbf{x})$.
There is a linear relationship between $\mathbf{y}$ and $\mathbf{x}$.

[Can check properties of linear relationship using $T(\mathbf{x}) = A(\mathbf{x})$]

[A linear law relates inputs and outputs of chemical reaction]

After 2 minutes the amounts are $\mathbf{z} = A^2\mathbf{x}$. After k minutes the amounts are $A^k\mathbf{x}$.

Have linear transformation from $\mathbf{y}$ to $\mathbf{x}$. A linear transformation comes from a linear system of equations in this case.

Definition 1: A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ another vector $T(\mathbf{x})$. The set $\mathbb{R}^n$ is called the *domain* of T and the set $\mathbb{R}^m$ is called the *codomain* of T.

Notation: $T : \mathbb{R}^n \to \mathbb{R}^m$.

Definition 2: The vector $T(\mathbf{x})$ is called the *image* of $\mathbf{x}$. The set of all images $T(\mathbf{x})$ is called the *range* of T.

Observation: A general linear transformation from a vector $\mathbf{x}$ to a vector $\mathbf{y}$ can always be written out as a system of equations where the x_i are linear combinations of the y_i (no squares, etc.); to be proved later.

Example 1: Multiplication by any matrix is linear in $\mathbb{R}^n$ (check properties).

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$

$$A(c\mathbf{x}) = cA\mathbf{x}.$$

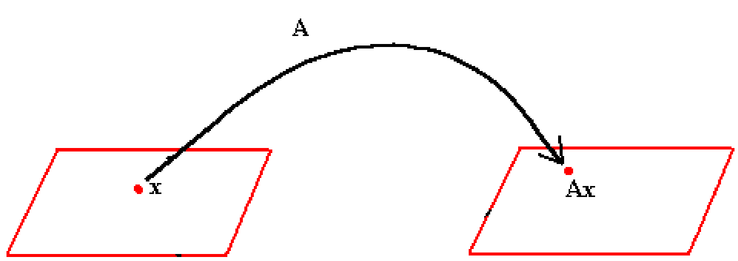


Figure: A viewed as a linear transformation: $A\mathbf{x} = T(\mathbf{x})$.

Fact (to be proved later)**:** *For every linear transformation $T(x)$, there exists a unique matrix A such that $T(x) = Ax$.*

A is called the *standard matrix* of $T(x)$.

Definition 3:

Example: American Automotive market - three major kinds of cars: American, Japanese, European. Every 5 years (average time between car purchases),

American market keeps 80% of previous buyers, and gets 20% of Japanese market, and 10% of European market
Japanese market keeps 70% of previous buyers, gets 10% of American market, and gets 30% of European market
European market keeps 60% of previous buyers, and gets 10% of American and 10% of Japanese market

Assuming non-steady state model, what are the percentage shares of the market?

$A:\ x_1 =$ number of American car owners (in millions)
$x_2 =$ number of Japanese car owners
$x_3 =$ number of European car owners

Then:

$y_1 =$ number of American cars in 5 years
$= .8x_1 + .2x_2 + .1x_3$

$y_2 =$ number of Japanese cars in 5 years
$= .1x_1 + .7x_2 + .3x_3$

$y_3 =$ number of European cars in 5 years
$= .1x_1 + .1x_2 + .6x_3$

$$\mathbf{y} = \begin{bmatrix} .8 & .2 & .1 \\ .1 & .7 & .3 \\ .1 & .1 & .6 \end{bmatrix} \mathbf{x}$$

again the input-output mapping is linear for this system.

Can again iterate system by squaring matrix, etc.

Domain: $\mathbb{R}^3$. Codomain: $\mathbb{R}^3$.

[later on vectors will have more general meanings; in that case we have:]

Example: Differentiation of polynomials is also a linear input-output map.

$L(P(x)) = \frac{d}{dx}P(x)$ is linear:

$$L(P_1(x) + P_2(x)) = \frac{d}{dx}(P_1(x) + P_2(x)) = \frac{d}{dx}P_1(x) + \frac{d}{dx}P_2(x)$$

$$= L(P_1(x)) + L(P_2(x))$$

$$L(cP(x)) = \frac{d}{dx}(cP(x)) = c\,\frac{d}{dx}\,P(x) = cL(P(x)).$$

3. Matrix of a Linear Transformation

Def. 4: The unit vector $\mathbf{e}_j$ in $\mathbb{R}^n$ is

$$\mathbf{e}_j \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with 1 in the j^{th} entry, and zeroes elsewhere.

Theorem 4: *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x}$$

$\forall\, \mathbf{x} \in \mathbb{R}^n$.

Furthermore, the j^{th} column of A is just the vector $T(\mathbf{e}_j)$, so that

$$A = [T(\mathbf{e}_1)\; T(\mathbf{e}_2) \ldots\; T(\mathbf{e}_n)]$$

Proof: The short proof of this is in the book.

[We now cover material from section 2.1]

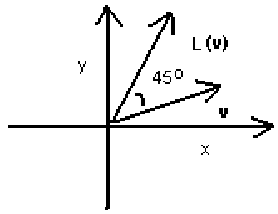
Example: Do matrix for rotation in two dimensions.

In space shuttle, a particle for experimental accelerometer is moving at 5 meters per second in direction $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. What is the velocity vector for someone in a rotated coordinate system by clockwise angle θ?

[How does someone rotate with respect to what you see in that vector?]

Note that vector will be rotated counterclockwise by θ.

First: counterclockwise rotation by $\pi/4 = 45^o$:



Note that if T is the operation which rotates vectors counterclockwise by 45^o, then:

$$T(\mathbf{v}_1 + \mathbf{v}_2) = \text{ rotation of } (\mathbf{v}_1 + \mathbf{v}_2)$$

$$= \text{ rotation of } \mathbf{v}_1 + \text{rotation of } \mathbf{v}_2$$

$$= T(\mathbf{v}_1) + T(\mathbf{v}_2)$$

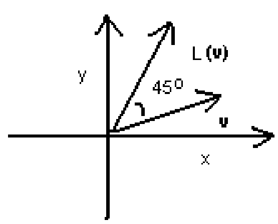
Similarly, show that $T(c\mathbf{v}) = cT(\mathbf{v})$;

Conclude: T is linear.

Use fact that any linear transformation on a finite dimensional space is given by a matrix. Thus there exists a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} = \text{ rotation of } \mathbf{x} \text{ by } 45^o \text{ counterclockwise}$$

To find A, note that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$\Rightarrow \qquad a = c = \sqrt{2}/2$

Similarly, get $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

$$\Rightarrow \qquad \begin{aligned} b &= -\sqrt{2}/2 \\ d &= \sqrt{2}/2 \end{aligned}$$

$$\Rightarrow \quad A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & +\sqrt{2}/2 \end{bmatrix}$$

Generally if A rotates by θ, can let:

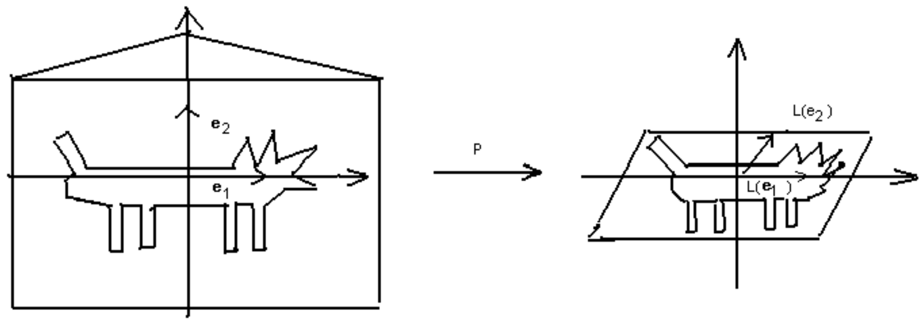
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where the entries are determined by action of A on $\mathbf{e}_1$ and $\mathbf{e}_2$

Preliminary: We will see later that if $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = *identity matrix*, then for any matrix A, we have $AI = IA = A$.

Ex: Look at the shear transformation (to be discussed in more detail later) from graphics:

Operation $P =$ perspective operation



What is the transformation which takes picture 1 to picture 2 ?

Answer: It is a linear transformation T (called P above)

Just compute $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, and all else is determined. Assume that

$$T(\mathbf{e}_1) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}; \qquad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus the matrix for this transformation is A where:

$$A = AI = A[\mathbf{e}_1, \mathbf{e}_2] = [A\mathbf{e}_1 A\mathbf{e}_2] = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix} = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)].$$

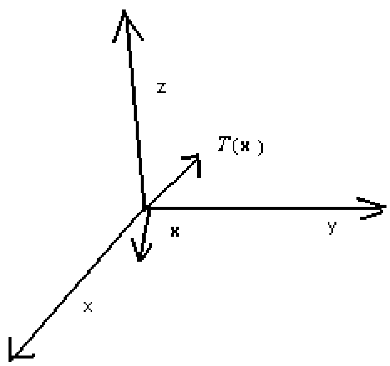
[this is how we transform points using linear transformations]

Recall the linear transformation $T(\mathbf{v})$ on the vector space $\mathbb{R}^2$ above. How do we find the matrix A of T? In above example,

$$A = [T(\mathbf{e}_1)\, T(\mathbf{e}_2)]$$

This is true in general with the same proof:

Example: Let T be the linear transformation which rotates vectors in $\mathbb{R}^3$ by 90 degrees counterclockwise about the x axis. What is the matrix A of T?



$T(\mathbf{e}_1) = \mathbf{e}_1$

$T(\mathbf{e}_2) = \mathbf{e}_3$

$T(\mathbf{e}_3) = -\mathbf{e}_2$

Therefore

$$A = [T(\mathbf{e}_1)\, T(\mathbf{e}_2)\; T(\mathbf{e}_3)] = [\mathbf{e}_1 \;\; \mathbf{e}_3 \; -\mathbf{e}_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus rotation of vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ counterclockwise about $\mathbf{x}$ axis gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

4. Geometric Linear Transformations

[Material through the end of this lecture will be elaborated on the board]

[The above rotation moves vectors; equivalently it moves points]

Consider the rotatation with matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

[90^o counterclockwise rotation]

Image of the unit square:

[What is the image of unit square under the above rotation?]

Other transformations:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

[reflection about x axis]

[Note image of unit square under the rotation]

2. $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

[Stretch by factor k horizontally on the unit square; note effect on $\mathbf{e}_1$ and $\mathbf{e}_2$]

3. $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

[Shear transformation; again note effect on unit vectors]

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

[Projection onto x axis]

5. Domain and Range

Recall the definitions of the domain and range of a LT T.

[We first want to consider some intuition behind the notions of one to one and onto]

[Now the formal definitions]

Def. 5:
A linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is *one to one* if $T(x)=T(y) \;\Rightarrow\; x=y$

A linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is *onto* if $\forall\; y \in \mathbb{R}^m\; \exists\, x \in \mathbb{R}^n$ s.t.

$$T(x)=y.$$

Theorem 5: *$T:\mathbb{R}^n \to \mathbb{R}^m$ is one to one iff the equation*

$$T(\mathbf{x})=\mathbf{0}$$

has only the trivial solution $\mathbf{x}=0$.

Equivalently, T is one to one if its matrix A has a pivot position in every column.
Proof: In book

Theorem 6: *The mapping T is onto iff the equation*

$$T(\mathbf{x})=\mathbf{b}$$

has a solution for all **b.**

Equivalently, T is onto if its matrix A has a pivot entry in every row.

Proof: In book

Example 2:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}.$$