15 Meyer and Daubechies Wavelets

1. Meyer wavelets: another example -

Scaling function:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \begin{cases} 1 \quad \text{if} \quad |\omega| \le 2\pi/3 \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega| - 1\right)\right] \quad \text{if} \quad 2\pi/3 \le |\omega| \le 4\pi/3 \\ 0 \quad \text{otherwise} \end{cases}$$

[error in Daubechies : $3/4\pi$ instead of $3/2\pi$ inside ν]

where ν is any infinitely differentiable non-negative function satisfying

$\nu(x) =$

$$\begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1 \\ \text{smooth transition in } \nu \text{ from } 0 \text{ to } 1 \text{ as } x \text{ goes from } 0 \text{ to } 1 \end{cases}$$

and

$$\nu(x) + \nu(1 - x) = 1.$$

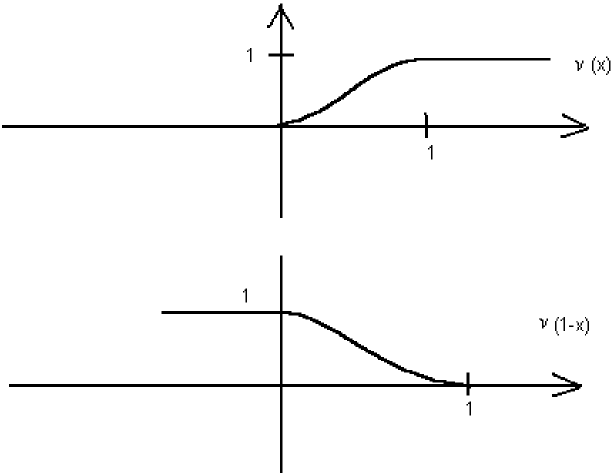


fig 34: $\nu(x)$ and $\nu(1-x)$

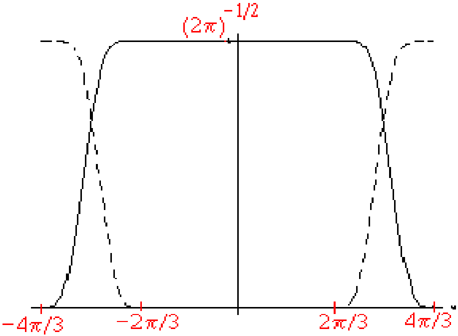


fig 35: Fourier transform $\widehat{\phi}(\omega)$ of the Meyer scaling function

Need to verify necessary properties for a scaling function:

(i)

$$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi} \qquad (21)$$

To see this, consider the two possible ranges of values of ω:

(a) $|\omega + 2\pi k_1| \le 2\pi/3$ for some k_1. In that case (see diagram above):

$$\widehat{\phi}(\omega + 2\pi k_1) = \frac{1}{\sqrt{2\pi}}; \qquad \widehat{\phi}(\omega + 2\pi k) = 0 \text{ if } k \neq k_1$$

since if $|\omega + 2\pi k_1| \le 2\pi/3$, then $|\omega + 2\pi k| \ge 4\pi/3$ for $k \neq k_1$. Thus (21) holds because there is only one non-zero term in that sum.

(b) $2\pi/3 \leq \omega + 2\pi k_1 \leq 4\pi/3$ for some k_1. In this case we also have

$$-4\pi/3 \leq \omega + 2\pi(k_1 - 1) \leq -2\pi/3.$$

Also, for all values $k \neq k_1$ or $k_1 - 1$, can calculate that

$$2\pi k \notin [-4\pi/3, 4\pi/3],$$

so

$$\widehat{\phi}(\omega + 2\pi k) = 0.$$

So sum has only two non-zero terms:

$$
\begin{aligned}
2\pi\sum_k|\widehat{\phi}(\omega+2\pi k)|^2 &= 2\pi\Big(|\widehat{\phi}(\omega+2\pi k_1)|^2+|\widehat{\phi}(\omega+2\pi(k_1-1)|^2\Big).\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega+2\pi k_1|-1\right)\right]+\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega+2\pi(k_1-1)|-1\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]+\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}(-(\omega+2\pi(k_1-1)))-1\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]+\cos^2\left[\frac{\pi}{2}\nu\left(-\frac{3}{2\pi}\omega-3k_1+2\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]+\cos^2\left[\frac{\pi}{2}\left(1-\nu\left(1-\left(-\frac{3}{2\pi}\omega-3k_1+2\right)\right)\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]+\cos^2\left[\frac{\pi}{2}-\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]+\sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]\\
&= 1
\end{aligned}
$$

Note that above $|\omega + 2\pi(k_1 - 1)| = -(\omega + 2\pi(k_1 - 1))$, since quantity in parentheses always negative for our range of ω. In next to last equality have used $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Note since cases (a), (b) cover all possibilities for ω (since they cover a range of size 2π for $\omega + 2\pi k_1$), we are finished proving (21).

Also need to verify:

(ii)

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2)$$

for some 2π-periodic $m_0(\omega/2)$. Indeed, looking at pictures:

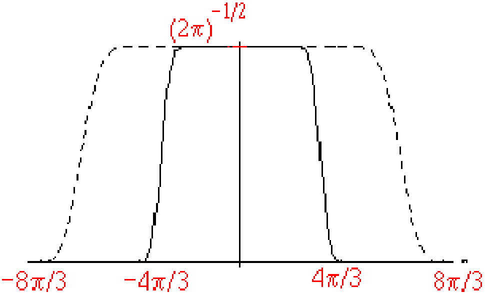


fig 36: $\widehat{\phi}(\omega)$ and $\widehat{\phi}(\omega/2)$ (----)

ratio of these two looks like:

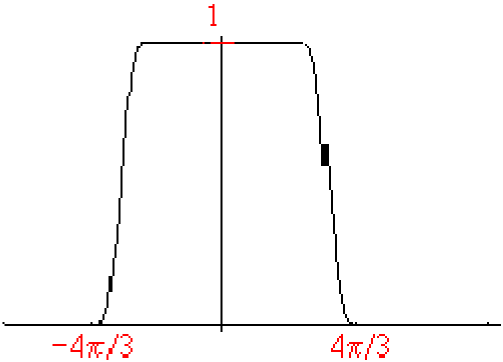


fig. 37: $\widehat{\phi(\omega)}/\widehat{\phi}(\omega/2) = \sqrt{2\pi}\,\widehat{\phi}(\omega)$ in the interval $[-2\pi, 2\pi]$.

Note since ratio $\widehat{\phi}(\omega)/\widehat{\phi}(\omega/2) = \sqrt{2\pi}\,\widehat{\phi}(\omega)$ in $[-2\pi, 2\pi]$, we can define

$$m_0(\omega/2) = \frac{\widehat{\phi}(\omega)}{\widehat{\phi}(\omega/2)} = \sqrt{2\pi}\,\widehat{\phi}(\omega) \tag{22}$$

if $\omega \in [-2\pi, 2\pi]$.

Definition ambiguous when numerator and denominator are 0.

Definition also ambiguous for $\omega \notin [-2\pi, 2\pi]$ since numerator and denominator both 0. So define $m_0(\omega/2)$ by periodic extension of above for all real ω.

How to do that? Just add all possible translates of the bump $\widehat{\phi}(\omega)$ to make it 4π-periodic:

$$m_0(\omega/2) = \sqrt{2\pi} \sum_k \widehat{\phi}\,(\omega + 4\pi k).$$

Check:

$$m_0(\omega/2)\widehat{\phi}(\omega/2) = \sqrt{2\pi}\sum_k \widehat{\phi}(\omega + 4\pi k)\widehat{\phi}(\omega/2)$$

$$= \sqrt{2\pi}\,\widehat{\phi}(\omega)\widehat{\phi}(\omega/2)$$

$$= \widehat{\phi}(\omega)$$

where we have used the fact that $\widehat{\phi}(\omega + 4\pi k)$ has no overlap with $\widehat{\phi}(\omega/2)$ if $k \neq 0$.
[So we expect a full MRA.]

2. Construction of the Meyer wavelet

Standard construction:

$$\widehat{\psi}(\omega) = e^{i\omega/2}\,\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2)$$

$$= e^{i\omega/2}\sum_k \overline{\widehat{\phi}(\omega+2\pi(2k+1))}\,\widehat{\phi}(\omega/2)$$

$$= e^{i\omega/2}\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2)$$

[supports of 2d and 3d factors do not overlap for other values of k; note $\overline{\widehat{\phi}} = \widehat{\phi}$ since $\widehat{\phi}$ is real]

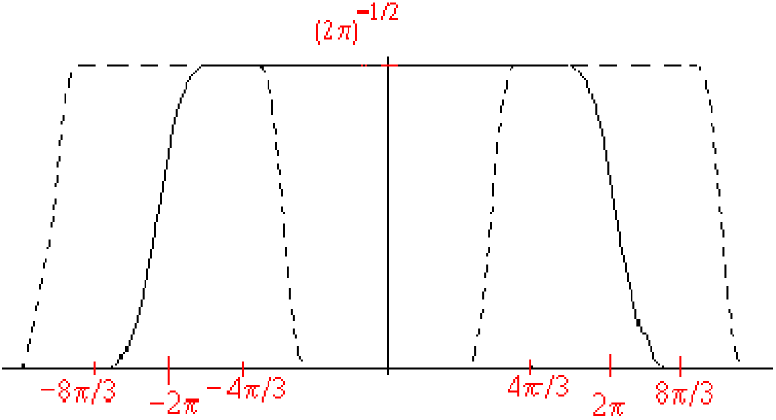


fig 38: $\widehat{\phi}(\omega + 2\pi) + \widehat{\phi}(\omega - 2\pi)$ (dashed) and $\widehat{\phi}(\omega/2)$

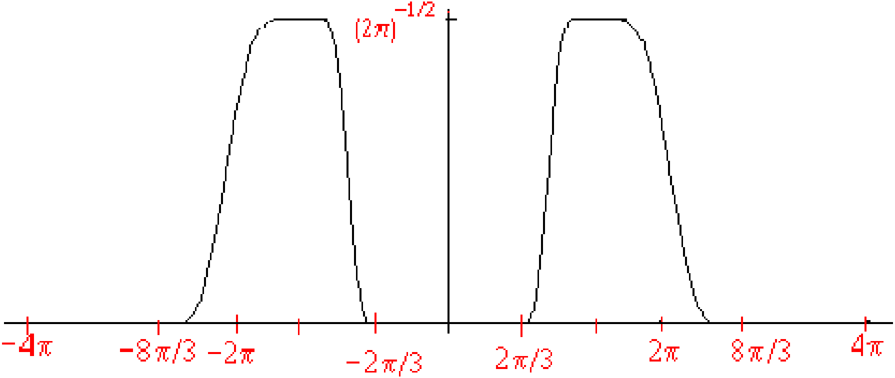


fig 39: $\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2)$

Thus have 2 distinct regions:

(a) For $2\pi/3 \le \omega \le 4\pi/3$ we see in diagram that

$$
\begin{aligned}
e^{-i\omega/2}\widehat{\psi}(\omega) &= \sqrt{2\pi}\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2) \\
&= \widehat{\phi}(\omega-2\pi) \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega-2\pi|-1\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(-\frac{3}{2\pi}(\omega-2\pi)-1\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(-\frac{3}{2\pi}\omega+2\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\left[1-\nu\left(1-\left(-\frac{3}{2\pi}\omega+2\right)\right)\right]\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\left[1-\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\right]
\end{aligned}
$$

$$= \frac{1}{\sqrt{2\pi}} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega - 1\right)\right]$$

So by symmetry same is true in $-2\pi/3 \leq \omega \leq \ -4\pi/3$, so replace ω by $|\omega|$ above to get:

$$e^{-i\omega/2}\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega| - 1\right)\right]$$

for $2\pi/3 \leq |\omega| \leq 4\pi/3$

(b) For $4\pi/3 \leq \omega \leq 8\pi/3$, we see from diagram (note $2\pi/3 \leq \omega/2 \leq 4\pi/3$):

$$
\begin{aligned}
e^{-i\omega/2}\widehat{\psi}(\omega) &= \sqrt{2\pi}\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2) \\
&= \widehat{\phi}(\omega/2) \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega/2-1\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}\omega-1\right)\right]
\end{aligned}
$$

Again by symmetry same is true in $-8\pi/3 \le \omega \le -4\pi/3$, so replace ω by $|\omega|$:

$$e^{-i\omega/2}\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\omega| - 1\right)\right]$$

for $\quad 4\pi/3 \le |\omega| \le 8\pi/3$

Thus:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{i\omega/2}\sin\left[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega| - 1\right], & \text{if} \quad 2\pi/3 \le |\omega| \le 4\pi/3 \\ e^{i\omega/2}\cos\left[\frac{\pi}{2}\nu(\ \frac{3}{4\pi}|\omega| - 1)\right], & \text{if} \ \ 4\pi/3 \le |\omega| \le 8\pi/3 \\ 0 \quad \text{otherwise} \end{cases}$$

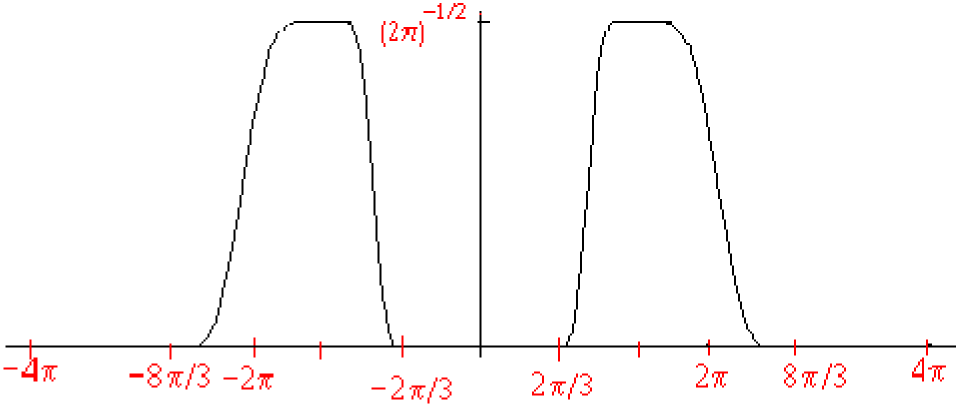


Fig. 40: The wavelet Fourier transform $|\widehat{\psi}(\omega)|$

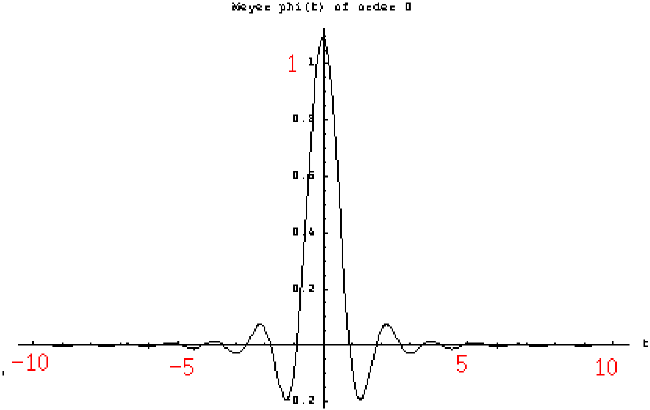


Fig. 41: The Meyer wavelet $\psi(x)$

3. Properties of the Meyer wavelet

Note: If ν is chosen as above and has all derivatives 0 at 0 and 1, can check that $\widehat{\psi}(\omega)$ is:

- infinitely differentiable (since it is a composition of infinitely differentiable functions), and one can check that all derivatives are 0 from both sides at the break. For example, the derivatives coming in from the left at $\omega = \frac{2\pi}{3}$ are:

$$\frac{d^n}{d\omega^n} \widehat{\psi}(\omega^-) \bigg|_{\omega = \frac{2\pi}{3}} = 0$$

and similarly

$$\frac{d^n}{d\omega^n}\widehat{\psi}(\omega^+)\Big|_{\omega=\frac{2\pi}{3}} = 0$$

(proof in exercises).

- supported (non-zero) on a finite interval

Lemma:

(a) If a function $\psi(x)$ has n derivatives which are integrable, then the Fourier transform satisfies

$$|\widehat{\psi}(\omega)| \leq K(1 + |\omega|)^{-n}. \tag{23}$$

Conversely, if (23) holds, then $\psi(x)$ has at least $n-2$ derivatives.

(b) Equivalently, if $\widehat{\psi}(\omega)$ has n integrable derivatives, then

$$|\psi(x)| \leq K(1 + |x|)^{-n} \tag{24}$$

Conversely, if (24) holds, then $\widehat{\psi}(\omega)$ has at least $n-2$ derivatives.

Proof: in exercises.

Thus: $\psi(x)$

- Decays at ∞ faster than any inverse power of x
- Is infinitely differentiable

Claim:

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$$

form an orthonormal basis for $L^2(\mathbb{R})$.

- Check (only to verify above results - we already know this to be true from our theory):

$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|^2\,d\omega = 1$$

Pf:

$$\int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|^2 \; d\omega = \frac{1}{2\pi}\left(\int_{\frac{2\pi}{3}\le|\omega|\le\frac{4\pi}{3}} d\omega \, \sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega|-1\right)\right] + \int_{\frac{4\pi}{3}\le|\omega|\le\frac{8\pi}{3}} d\omega \, \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\omega|-1\right)\right]\right)$$

[getting rid of the $|\cdot|$ and doubling; changing vars. in second integral]

$$= \frac{1}{\pi}\left(\int_{\frac{2\pi}{3}\leq\omega\leq\frac{4\pi}{3}} d\omega\, \sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega - 1\right)\right]\right.$$

$$\left. + 2\int_{\frac{2\pi}{3}\leq\omega\leq\frac{4\pi}{3}} d\omega\, \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega - 1\right)\right]\right)$$

$$= \frac{1}{\pi}\left(\int_{\frac{2\pi}{3}\leq\omega\leq\frac{4\pi}{3}} d\omega\, \left\{\sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega - 1\right)\right] + 2\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega - 1\right)\right]\right\}\right)$$

$$= \frac{1}{\pi}\left(\int_{\frac{2\pi}{3}\leq\omega\leq\frac{4\pi}{3}} d\omega \left\{1 + \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega - 1\right)\right]\right\}\right)$$

[letting $s = \frac{3}{2\pi}\omega - 1 \;\Rightarrow\; \omega = 2\pi/3(s+1)$]

$$= \frac{2}{3}\left(\int_0^1 ds \left(1 + \cos^2\left[\frac{\pi}{2}\nu(s)\right]\right)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds \left(1 + \cos^2\left[\frac{\pi}{2}\nu(s)\right]\right) + \int_{1/2}^{1} ds \left(1 + \cos^2\left[\frac{\pi}{2}\nu(s)\right]\right))\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds \left(1 + \cos^2\left[\frac{\pi}{2}\nu(s)\right]\right) + \int_0^{1/2} ds \left(1 + \cos^2\left[\frac{\pi}{2}\nu(s+1/2)\right]\right)\right)$$

[using $\nu(s+1/2) = 1-\nu(1/2-s)$]

$$= \frac{2}{3}\left(\int_0^{1/2} ds\left(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\right) + \int_0^{1/2} ds\left(1+\cos^2\left[\frac{\pi}{2}(1-\nu(1/2-s))\right]\right)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds\left(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\right) + \int_0^{1/2} ds\left(1+\sin^2\left[\frac{\pi}{2}\nu(1/2-s)\right]\right)\right)$$

$$\overset{s\to 1/2-s}{=} \frac{2}{3}\left(\int_0^{1/2} ds\left(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\right) + \int_0^{1/2} ds\left(1+\sin^2\left[\frac{\pi}{2}\nu(s)\right]\right)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds(2+1)\right) = 1$$

- To show in another way that they form an orthonormal basis, sufficient to show that for arbitrary $f \in L^2(\mathbb{R})$,

$$\sum_{j,k}^{\infty} |\langle \psi_{jk}, f \rangle|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

[this is a basic analytic theorem].

Now note:

$$\sum_{j,k}^{\infty} |\langle \psi_{jk}, f \rangle|^2 = \sum_{j,k}^{\infty} \left| \int dx \, \overline{\psi_{jk}(x)} f(x) dx \right|^2$$

$$= \sum_{j,k}^{\infty} \left| \int d\omega \, \widehat{f}(\omega) \overline{\widehat{\psi}_{jk}(\omega)} \right|^2 .$$

Note if

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k).$$

Then as usual:

$$\widehat{\psi}_{jk}(\omega) = 2^{-j/2}\,\widehat{\psi}(2^{-j}\omega)\,e^{-i2^{-j}k\omega}.$$

Plug this in above and can do calculation to show (we won't do the calculation):

$$\sum_{j,k}^{\infty} |\langle f, \psi_{jk}\rangle|^2 = \int_{-\infty}^{\infty} dx\,|f(x)|^2\,,$$

as desired.

CONCLUSION:

The wavelets

$$\psi_{jk}(x) \equiv 2^{j/2}\,\psi(2^j x - k)$$

form an orthonormal basis for the square integrable functions on the real line.

4. Daubechies wavelets:

Recall that one way we have defined wavelets is by starting with the scaling (pixel) function $\widehat{\phi}(x)$. Recall it satisfies:

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}\,(\omega/2)$$

for all ω, where $m_0(\omega)$ is some periodic function. If we use m_0 as the starting point, recall we can write

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty} m_0(\omega/2^j). \quad (25)$$

Recall m_0 is periodic, and so has Fourier series:

$$m_0(\omega) = \sum_k a_k \, e^{-ik\omega} \, .$$

If m_0 satisfies $|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$, then it is a candidate for construction of wavelets and scaling functions.

For Haar wavelets, recall $m_0(\omega) = e^{i\omega/2} \cos \; \omega/2,$ so we could plug into (25) to get $\widehat{\phi}$, and then use previous formulas to get wavelet $\psi(x)$.

If we *start* with a function $m_0(\omega)$, when does (25) lead to a genuine wavelet? Check conditions:

(1)

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j)$$

$$= \frac{1}{\sqrt{2\pi}} m_0(\omega/2) \prod_{j=2}^{\infty} m_0(\omega/2^j)$$

$$= m_0(\omega/2)\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty} m_0(\omega/2^{j+1})$$

$$= m_0(\omega/2)\widehat{\phi}(\omega/2) \tag{26}$$

Recall this implies that $V_j \subset V_{j+1}$ where

$$V_j = \left\{ \sum_{k=-\infty}^{\infty} a_k \phi_{jk}(x) \middle| \sum_k |a_k|^2 < \infty \right\}$$

(usual definition) with $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$

(2) The second condition we need to check is that translates of ϕ orthonormal, i.e.,

$$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}.$$

If

$$m_0(\omega) = \text{finite Fourier series}$$

$$= \sum_{k=-N}^{N} a_k e^{-i\omega k} = \text{trigonometric polynomial}$$

There is a simple condition which guarantees condition (2) holds.

Theorem (Cohen, 1990): If the trigonometric polynomial m_0 satisfies $m_0(0) = 1$ and

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1 \qquad (27)$$

(our standard condition on m_0), and also $m_0(\omega) \neq 0$ for $|\omega| \leq \pi/3$, then condition (2) above is satisfied by

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j)$$

Proof: Daubechies, Chapter 6.

Since condition **(1)** is also automatically satisfied, this means ϕ is a scaling function which will lead to a full orthonormal basis using our algorithm for constructing wavelets.

Another choice of m_0 is:

$$m_0(\omega) = \frac{1}{8}[(1+\sqrt{3}) + (3+\sqrt{3})e^{-i\omega} + (3-\sqrt{3})e^{-2i\omega} + (1-\sqrt{3})\, e^{-3i\omega}]$$

(Fourier series with finite number of terms).

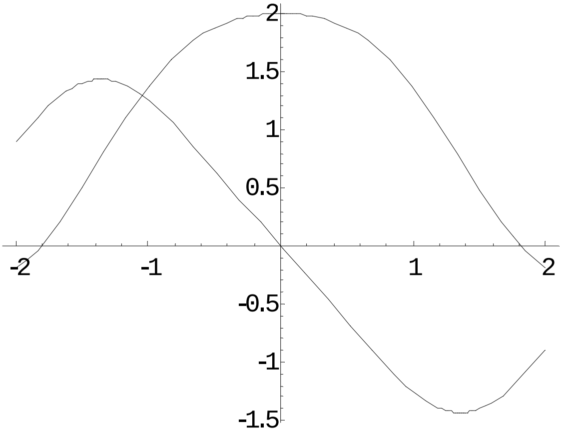


Fig 42: Real (symmetric) and imaginary (antisymmetric) parts of $m_0(\omega)$

To check Cohen's theorem satisfied:

(i) Equation (27) satisfied (see exercises).

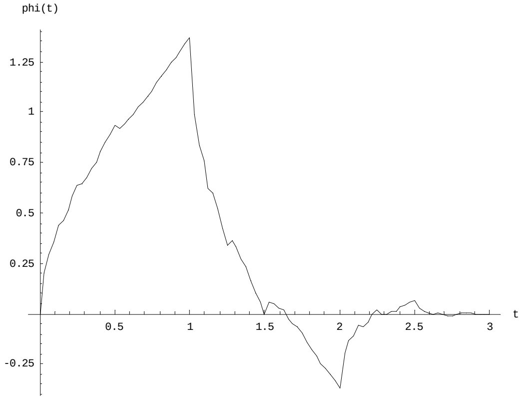
(ii) If $m_0(\omega) = \text{Re}\, m_0(\omega) + i\ \text{Im}\, m_0(\omega)$,

$$|m_0(\omega)|^2 = |\text{Re}\ m_0(\omega)|^2 + |\text{Im}\, m_0(\omega)|^2 \neq 0$$

for $|\omega| \leq \pi/3$, as can be seen from graph above.

So: conditions of Cohen's theorem are satisfied.

In this case if we define scaling function ϕ by computing infinite product (25) (perhaps numerically), and then use our standard procedure to construct wavelet $\psi(x)$, we get:



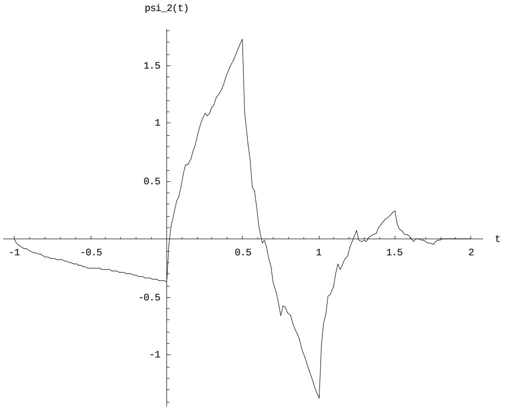


fig 43: pictures of ϕ and ψ

Note meaning of m_0: In terms of the original wavelet, this states

$$\phi(x) = \frac{1}{4}\left[(1+\sqrt{3})\phi(2x) + (3+\sqrt{3})\phi(2x-1)\right.$$

$$\left. + (3-\sqrt{3})\phi(2x-2) + (1-\sqrt{3})\phi(2x-3)\right]$$

(see (26) above). Note this equation gives the information we need on ϕ, since it determines $m_0(\omega)$.