# 16 Additional Topics: Compact and Continuous Wavelet expansions

## 1. Other examples

Note again it is possible to get other wavelets this way: If we demand

$$\phi(x) = .226\,\phi(2x) + .854\,\phi(2x-1) + 1.24\,\phi(2x-2)$$

$$+ .196\,\phi(2x-3) - 1.434\,\phi(2x-4) - .046\,\phi(2x-5)$$
$$+ .110\,\phi(2x-6) - .008\,\phi(2x-7) - .018\,\phi(2x-8)$$
$$+ .004\,\phi(2x-9) \qquad (28)$$

Then this results with an $m_0(\omega)$

$$m_0(\omega) = .113 + .427\, e^{i\omega} + .512\, e^{2i\omega} + .098 e^{3i\omega} + \ldots + .002 e^{9i\omega}.$$

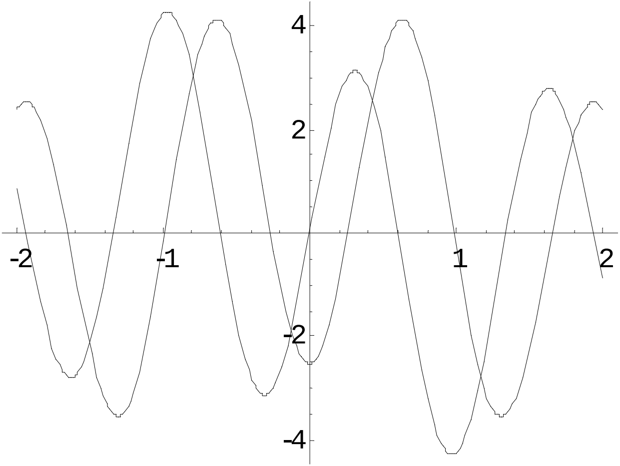



Fig 44: Real (symmetric) and imaginary parts of $m_0$; note condition (ii) of Cohen's theorem is satisfied.

Can check it satisfies condition (ii) of Cohen's theorem and resulting $\phi$ is obtained:

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} m_0(\omega/2^j).$$

It satisfies required properties (a) - (f) of a multiresolution analysis. Corresponding scaling function ${}_5\phi(x)$ and wavelet ${}_5\psi(x)$ are below

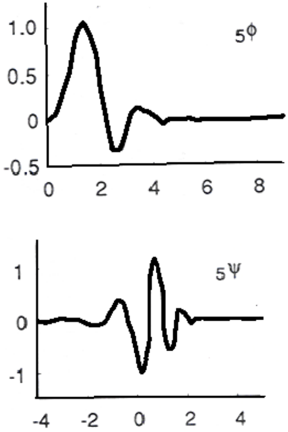



Fig 45: Scaling function and wavelet for the above $\phi$ choice

NOTE: Can show that if there is a finite number of terms on the right side of (28), then corresponding wavelet and scaling function are compactly supported.

## 2. Numerical uses of wavelets

Note that once we have an orthonormal wavelet basis $\{\psi_{jk}\}$, can write any function:

$$f(x) = \sum_{j,k} a_{jk}\, \psi_{jk}(x),$$

with $a_{jk} = (f, \psi_{jk})$. Numerically, can find $a_{jk} = \langle \psi_{jk}, f \rangle$ using numerical integration to evaluate inner product.

With Daubechies and other wavelets, there are no closed form for the wavelets, so above integrations must be performed on the computer.

But there are very efficient methods of doing this: in order to get *all* the wavelets $\psi_{jk}$ into the computer, we just need to input one - all others are rescalings and translations of the original one.

There are efficient algorithms to get coefficients $a_{jk}$; more details in Daubechies' book.

## 3. SOME GENERAL PROPERTIES OF ORTHONORMAL WAVELET BASES:

**Theorem:** If the basic wavelet $\psi(x)$ has exponential decay, then $\psi$ cannot be infinitely differentiable.

(in particular, if $\psi$ has compact support, then $\psi$ cannot be infinitely differentiable).

Proof: Daubechies, Chapter 5.

## Compactly Supported Wavelets:

So far we are able to get wavelets

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$$

which form an orthonormal basis for $L^2$. Note Haar wavelets had compact support. When will wavelets be compactly supported in general?

Recall we assume that given basic scale space $V_0$, that we have scaling (pixel) function $\phi$ such that $\{\phi(x-k)\}_k$ form basis for $V_0$.

Recall

- $V_0 \subset V_1$ ,
- $\phi(x) \in V_0 \qquad \Rightarrow \qquad \phi(x) \in V_1$
- $\sqrt{2}\ \phi(2x) \in V_1$
- $\{\sqrt{2}\ \phi(2x-k)\}_{k=1}^{\infty}$ form a basis for $V_1$

Recall since $\phi(x) \in V_1$, we have for some choice of $h_k$:

$$\phi(x) = \sum_{0}^{\infty} h_k \sqrt{2}\ \phi(2x-k).$$

Constants $h_k$ relate the space $V_0$ to $V_1$.

We will see that:

**Theorem:**
Finitely many $h_k \neq 0 \quad \Leftrightarrow \quad \psi$, $\phi$ have compact support.

Proof:

$\Leftarrow$ : Assume $\phi$ has compact support. Then note since $\sqrt{2}\phi(2x-\ell)$ are orthonormal,

$$h_\ell = \int \sqrt{2}\phi(2x-\ell)\phi(x)dx$$

$= 0$ for all but a finite number of $\ell$ :

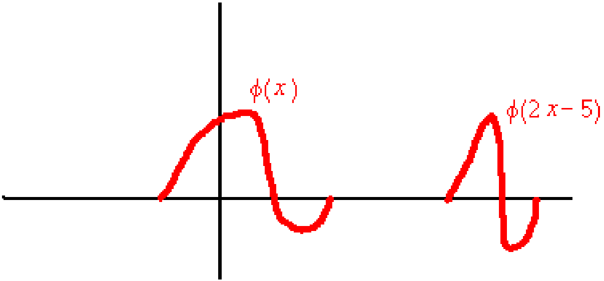



fig 46 : Note $h_l =$ integral of product $= 0$ for all but finite number of $\ell$ to prove $\Rightarrow$ : (rough sketch only)

Assume that $h_k$ are $0$ for all but a finite number of $k$.
Then need to show $\phi(x)$ has compact support.

Strategy of proof: look at $\widehat{\phi}(\omega)$.

Recall we defined

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega}$$

Recall:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(2^{-j}\omega).$$

- From this show that $\widehat{\phi}(\omega)$ extends to an analytic function of $\omega$ in whole complex plane satisfying:

$$|\widehat{\phi}(\omega)| \leq C(1 + |\omega|)^M \, e^{N|\mathrm{Im}\,\omega|}$$

for constants $M$ and $N$.

- This implies by Paley-Wiener type theorems that $\phi(x) = F^{-1}\left(\widehat{\phi}\right)$ is compactly supported. $\square$

## 4. GENERIC PRESCRIPTION FOR COMPACTLY SUPPORTED WAVELETS:

- Start with finite sequence of numbers $h_k$ (define how $V_0$ will be related to $V_1$)

- Construct

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega}$$

check that it satisfies Cohen's theorem conditions :

$|m_0(\omega)| \neq 0$ for $|\omega| \leq \pi/3.$

and

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1.$$

- Construct

$$\frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(2^{-j}\omega) = \widehat{\phi}(\omega)$$

- Construct Fourier transform of wavelet by:

$$\widehat{\psi}(\omega) \equiv e^{i\omega/2}\,\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2),$$

- Take inverse Fourier transform to get $\psi(x) =$ wavelet

## 5. SOME FURTHER PROPERTIES OF WAVELET EXPANSIONS

QUESTION: Do wavelet expansions actually converge to the function being expanded at individual points $x$?

Assume that scaling function $\phi$ is bounded by an integrable decreasing function. Then:

**Theorem:** If $f$ is a square integrable function, then the wavelet expansion of $f$

$$f(x) = \sum_{j,k}^{\infty} a_{jk}\, \psi_{jk}(x)$$

converges to the function $f$ almost everywhere (i.e., except on a set of measure $0$).

QUESTION: How fast do wavelet expansions converge to the function $f$?

ANSWER: That depends on how “regular" the wavelet $\psi$ is. More particularly it depends exactly on the Fourier transform of $\psi$:

**Theorem:** In $d$ dimensions, the wavelet expansion

$$f(x) = \sum_{j,k} a_{jk}\, \psi_{jk}(x)$$

converges to a smooth $f$ in such a way that the partial sum

$$\sum_{j \le N, k} a_{jk} \psi_{jk}(x)$$

differs from $f(x)$ at each $x$ by at most $C \cdot 2^{-Ns}$, iff

$$\int |\widehat{\psi}(\omega)|^2 \, |\omega|^{-2s-d} \, d\omega < \infty.$$

# 6. CONTINUOUS WAVELET TRANSFORMS

Consider a function $\psi(x) \in L^2$ (i.e., $\psi$ is square integrable), such that $\psi(x)$ decays fast enough at $\infty$ (faster than $1/x^2$), and such that

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0.$$

Then we can define an integral wavelet expansion (integrals instead of sums) using re-scalings of $\psi(x)$:

Define rescaled functions

$$\psi_{a,b}(x) \equiv |a|^{1/2}\,\psi(a(x-b)).$$

[note $a \to 1/a$ in definition of Daubechies]

Here $a, b \in \mathbb{R}$. Thus $a$ measures how much $\psi$ has been stretched (dilation parameter), and $b$ measures how much $\psi$ has been moved to the right (translation parameter).

New point: dilation parameter $a$ and translation parameter $b$ can take on any real value.

Now define wavelet expansions in this case (analogous to Fourier transform -- called wavelet transform): given $f \in L^2(\mathbb{R})$, we define the transform (assuming that $\psi$ is real)

$$(Wf)(a,b) = \int dx\, f(x)\, \overline{|a|^{1/2}\psi(a(x-b))}$$

$$= \int dx\, f(x)\, \overline{\psi_{a,b}(x)}$$

$$= \left\langle \psi_{a,b}, f \right\rangle$$

How to recover $f$ from $(Wf)(a,b)$?

Claim:

$$f(x) = C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} da\, db \ \ (Wf)(a,b)\, \psi_{a,b}(x)$$

where

$$C^{-1} = -2\pi \int d\omega\, |\omega|^{-1} |\widehat{\psi}(\omega)|^2.$$

Pf. of claim (sketch; details in Daubechies, Ch. 2):

We will show that for any $g(x) \in L^2$,

$$\langle g(x), f(x) \rangle =$$

$$\left\langle g(x), C \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da\, db\, (Wf)(a,b)\, \psi_{a,b}(x) \right\rangle$$

To see this, note that

$$\langle g(x), f(x) \rangle = \int_{-\infty}^{\infty} \overline{g(x)}\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} d\omega\; \overline{\widehat{g}\,(\omega)}\, \widehat{f}(\omega)$$

[next use “Plancherel Theorem” for wavelet transforms]

$$= C \int\int da\, db\, \overline{(W g)(a,b)}(W f)(a,b)$$

$$= C \int\int da\, db\, \langle g(x), \psi_{a,b}(x)\rangle (W f)(a,b)(x)$$

$$= \left\langle g(x), C \int\int da\, db\, (W f)(a,b)\psi_{a,b}(x) \right\rangle,$$

as desired, completing the proof.

Thus we know how to recover $f(x)$ from $Wf(a,b)$ (analogous to recovering $f(x)$ from $\widehat{f}(\omega)$ in Fourier transform).

QUESTION: What sorts of functions are $(Wf)(a,b)$? For some choices of $\psi$, these are spaces of analytic functions.

## 7. Convolutions:

**Definition:** The *convolution* of two functions $f(x)$ and $g(x)$ is defined to be

$$f(x)*g(x) \equiv \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

**Theorem 3:** The convolution is commutative: $f*g = g*f$

*Proof:* Exercise.

**Theorem 4:** The Fourier transform of a convolution is a product. Specifically,

$$\mathcal{F}(f(x)*g(x)) = \sqrt{2\pi}\widehat{f}(\omega)\widehat{g}(\omega)$$

*Proof:* Exercise.

**Lemma 5:** For any function $f$, $\mathcal{F}(f(-x)) = \overline{\widehat{f}(\omega)}$

*Proof:* Exercise.

## 8. APPLICATION OF INTEGRAL WAVELET TRANSFORM: IMAGE RECONSTRUCTION (S. Mallat)

Dyadic wavelet transform: a variation on continuous wavelet transform.

Now define new dilation only by powers of 2; arbitrary translations:

$$\psi_{j,b}(x) = 2^j \psi(2^j(x-b))$$

Define

$$\psi_j(x) = 2^j \psi(2^j x).$$

(Still allow $b \in \mathbb{R}$ to take all values, but restrict $a = 2^j$.)

Define this dyadic (partially discrete) wavelet transform by:

$$(Wf)(j,b) = \int f(x)\,\psi_{j,b}(x)\,dx$$

i.e., usual set of wavelet coefficients, except that $b$ is continuous.

Note:

$$(Wf)(j,b) = \int f(x)\, \psi_{j,b}(x)\, dx$$

$$= \int dx\, f(x)\, 2^j \psi(2^j(x-b))$$

$$= \int dx\, f(x)\, \psi_j(x-b)$$

$$= (f * \psi_j)(b)$$

(a convolution) where as above

$$\psi_j(x) = 2^j \psi(2^j x) = \text{ shrinking of } \psi \text{ by a factor } 2^j.$$

New assumption: Fourier transform $\widehat{\psi}(\omega)$ satisfies

$$\sum_{j=-\infty}^{\infty} |\widehat{\psi}(2^j \omega)|^2 = \frac{1}{2\pi}.$$

Now: given $f(x)$, consider dyadic wavelet transform; $a = 2^j$ only:

Can show under our assumptions that can recover $f$ in this case too:

Recovery formula for $f$ is:

$$f(x) = \sum_{j=-\infty}^{\infty} (Wf)(j,x) * \psi_j(-x)$$

(convolution in variable $x$). It is easy to check that this is correct: if $\mathcal{F}$ denotes Fourier transform:

$$\mathcal{F}\left( \sum_{j=-\infty}^{\infty} (Wf)(j,x) * \psi_j(-x) \right)$$

$$= \mathcal{F}\left( \sum_{j=-\infty}^{\infty} f(x) * \psi_j(x) * \psi_j(-x) \right)$$

$$= \sum_{j=-\infty}^{\infty} \mathcal{F}(f(x) * \psi_j(x) * \psi_j(-x))$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \widehat{f}(\omega)\, \widehat{\psi}_j(\omega)\, \overline{\widehat{\psi}_j(\omega)}$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \widehat{f}(\omega)\,\widehat{\psi}(2^{-j}\omega)\;\overline{\widehat{\psi}(2^{-j}\omega)}\,.$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \widehat{f}(\omega)|\widehat{\psi}(2^{-j}\,\omega)|^2$$

$$= \widehat{f}(\omega)\, 2\pi \sum_{j=-\infty}^{\infty} |\widehat{\psi}(2^{-j}\,\omega)|^2$$

$$= \widehat{f}(\omega).$$

QUESTION: Given $f(x)$, what sort of function is the wavelet transform $(Wf)(j,b)$, as a function of $j$ and $b$?

Let $V =$ the collection of possible functions $(Wf)(j,b) =$ collection of possible wavelet transforms. When is an arbitrary function $g(j,b)$ a wavelet transform?

Can check that $g$ must satisfy a so-called reproducing kernel equation: $g(j,b)$ is the wavelet transform of some function iff

$$g(j,b) = (\mathcal{K}g)(j,b) \equiv \sum_{\ell=-\infty}^{\infty} \psi_j(b) * \psi_\ell(-b) * g(\ell,b)$$

[this equation defines $\mathcal{K}g$; note convolution is in $b$.]

Back to recovering $f$ from wavelet transform:

Thus we can recover $f$ as a sum of $f$ at different scales:

$$f = \sum_{j=-\infty}^{\infty} (Wf)(j, x) * \psi_j(-x).$$

Since $\psi$ is a known function, we can recover $f$ from the sequence of functions. Assume $a(x)$ is a cubic B-spline:

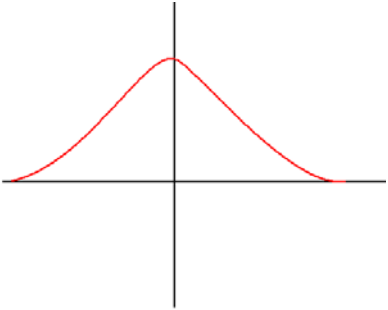


Fig. 47: A cubic B-spline $a(x)$ is a symmetric compactly supported piecewise cubic polynomial function whose transition points are twice continuously differentiable

Now let the wavelet be its first derivative: $\psi(x) = \frac{d}{dx}a(x)$

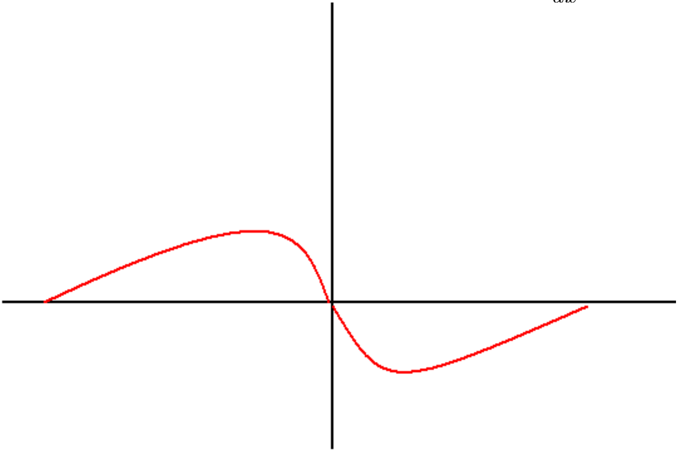
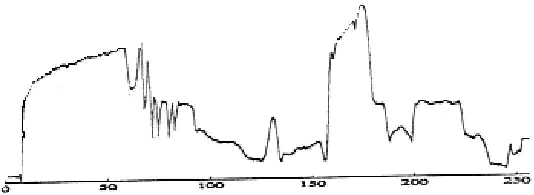


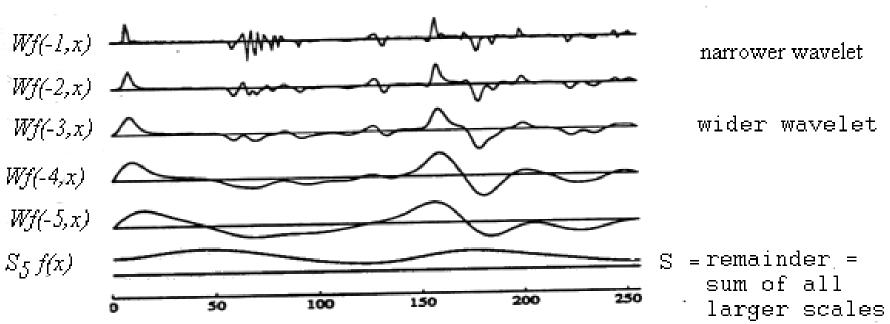
Fig 48: $\psi(x) = \frac{d}{dx}a(x)$ is the wavelet

Using the wavelet $\psi(x)$ :

$$
\begin{aligned}
&(Wf)(-2, x)\\
&(Wf)(-1, x)\\
&(Wf)(0, x)\\
&(Wf)(1, x)\\
&(Wf)(2, x)\\
&(Wf)(3, x)
\end{aligned}
$$

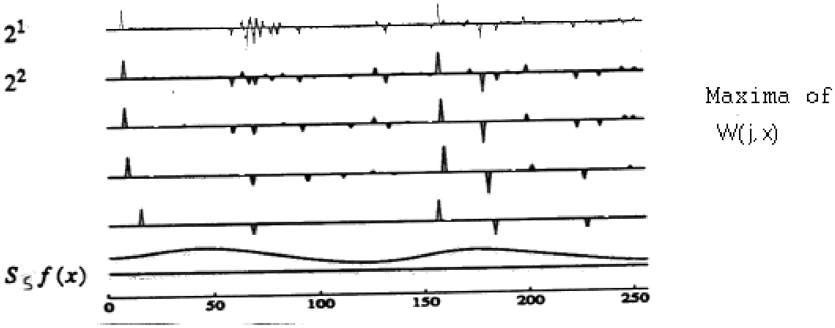
To see that these pieces of $f$ represent $f$ at different scales, look at example:


0
50
100
150
200
250



Wf(-1,x)
Wf(-2,x)
Wf(-3,x)
Wf(-4,x)
Wf(-5,x)
S5 f(x)
0
50
100
150
200
250
narrower wavelet
wider wavelet
S = remainder =
sum of all
larger scales

(b)




So: one can recover $f$ from knowing the functions

$$(Wf)(j,x).$$

This is a lot of functions. What advantage of storing $f$ in such a large number of functions? We can compress the data.

CONJECTURE: We can recover $f$ not from knowing all of the functions $W(j,x)$, but just from knowing their maxima and minima.

Meyer has proved this conjecture false strictly speaking certain choices of $\psi$ (including the above derivative $\psi(x)$ of the cubic spline). It has been proved true for another choice, the derivative of a Gaussian.

$$\psi(x) = \frac{d}{dx} e^{-x^2}$$

However, for either choice of $\psi$ numerically it is possible to recover $f(x)$ from knowing only the maxima and minima of the functions $W(j, x)$.

Numerical method:

Assume that we are given only the maxima and minima points of the function $W(j,x)$ for each $j$. How to recover $f$?

Given $f$, first take its wavelet transform; get $W(j,x)$.

Define

$\Gamma =$ set of all functions $g(j,x)$ which have the same set of maxima and minima (in $x$) as $W(j,x)$ for each $j$.

$V =$ set of all $g(j,x)$ which are wavelet transforms of some function of $x$.

Idea is: the true wavelet transform $Wf(j,x)$ of our given function $f(x)$ is in $\Gamma$ (i.e. has the same maxima as itself) and is in $V$ (i.e., in the collection of functions which are wavelet transforms).

Thus

$$W f \in \Gamma \cap V.$$

intuitive picture:

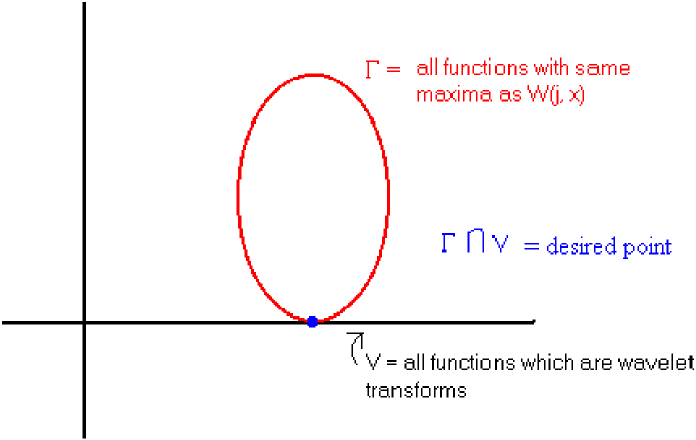



fig 49

Thus if we know just the maxima of $Wf(j,x)$, we can try to find $Wf(j,x)$

That is:

1. We know maxima of $Wf(j,x)$, so
2. know $\Gamma =$ all functions with same maxima as $Wf(j,x)$
3. Find $Wf(j,x)$ as “unique” point in $\Gamma$ which is also a wavelet transform, i.e., unique point in $\Gamma \cap V$:

**Algorithm:**

1. Start with only the maxima information about $W(j,x)$. Call $M$ the maxima information.

2. Make initial guess using function $g_1(j,x)$ which has the same maxima as $W(j,x)$.

3. Find closest function in $V =$ set of wavelet transforms to $g_1(j,x)$. Call this function $g_2(j,x)$.

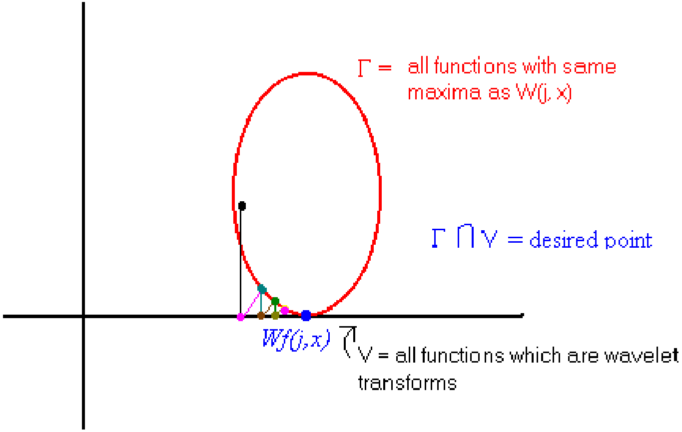
4. Find closest function in $\Gamma =$ functions with same maxima as $M$ to $g_2(j,x)$. Call this function $g_3(j,x)$.

5. Find closest function in $V$ to $g_3(j,x)$; call this $g_4(j,x)$.

6. Find closest function in $\Gamma$ to $g_4$; call this $g_5$.

7. Continue this way: at each stage $j$ find the closest function $g_j$ to $g_{j-1}$ in
the space $V$ or $\Gamma$ (alternatingly).

Eventually the $g_j(j,x) \underset{j\to\infty}{\longrightarrow} Wf(j,x)$ as desired.



CONCLUSION: We can recover the wavelet transform $Wf(j,x)$ of a function just by knowing its maxima in $x$.

THE POINT: Compression. We can store the maxima of $Wf$ using a lot less memory.

APPLICATION: Compression of images:

**Fig. 9:** *The upper left is the original lady image. The upper right image is a reconstruction from the maxima representation shown in the second column of fig. 8. This reconstruction is performed with 8 iterations and the noise to signal ratio is 6.6 $10^{-2}$. The lower left and lower right images have been reconstructed from the maxima representation shown respectively in the third and fourth column of fig. 8 (thresholding by the factors 4 and 8). The light textures have disappeared but the strong edges and textures remain unchanged.*

Fig. 50

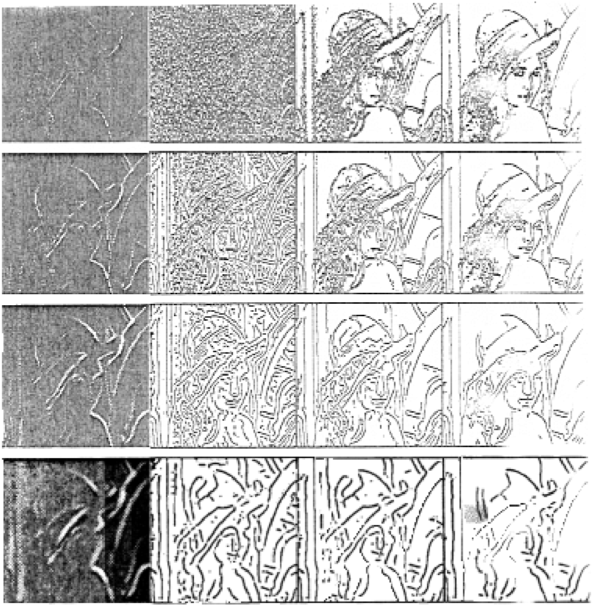


**Fig. 8:** *The first column gives the modulus images* $M_{2^j}f(x,y)$ *for* $1 \leq j \leq 4$ *of the lady image shown at the top left of fig. 9. The second column displays the position of the maxima of* $M_{2^j}f(x,y)$. *The third and fourth columns display the position of the local maxima whose amplitude are respectively larger than 4 and 8. The maxima that have been removed correspond essentially to the noise and the light texture irregularities.*

Fig. 51

## 9. Wavelets and Wavelet Transforms in Two Dimensions

Multiresolution analysis and wavelets can be generalized to higher dimensions. Usual choice for a two-dimensional scaling function or wavelet is a product of two one-dimensional functions. For example,

$$\phi_2(x, y) = \phi(x)\phi(y)$$

and scaling equation has form

$$\phi(x,y) = \sum_{k,l} h_{kl} \cdot 2\phi(2x-k, 2y-l).$$

Since $\phi(x)$ and $\phi(y)$ both satisfy the sclaing equation

$$\phi(x) = \sum_{k} h_k \cdot \sqrt{2}\phi(2x-k),$$

we have $h_{kl} = h_k h_l$. Thus two dimensional scaling equation is product of two one dimensional scaling equations.

We can proceed analogously to construct wavelets using products of one-dimensional functions. However, unlike one-dimensional case, we have three rather than one basic wavelet. They are:

$$\psi^{(I)}(x,y) = \phi(x)\psi(y)$$

$$\psi^{(II)}(x,y) = \psi(x)\phi(y)$$

$$\psi^{(III)}(x,y) = \psi(x)\psi(y).$$

The generalization of the one-dimensional wavelet equation leads to the following relations:

$$\psi^{(I)}(x,y) = \sum_{k,l} g_{kl}^{(I)} \cdot 2\phi(2x-k, 2y-l)$$

$$\psi^{(II)}(x,y) = \sum_{k,l} g_{kl}^{(II)} \cdot 2\phi(2x-k, 2y-l)$$

$$\psi^{(III)}(x,y) = \sum_{k,l} g_{kl}^{(III)} \cdot 2\phi(2x-k, 2y-l)$$

where $g_{kl}^{(I)} = h_k g_l,\quad g_{kl}^{(II)} = g_k h_l,$ and $g_{kl}^{(II)} = g_k g_l.$

We can generate two-dimensional scaling functions and wavelets using the functions ScalingFunction and Wavelet then taking the product. For example, here we plot the Haar wavelets in two dimensions. Various translated and dilated versions of the wavelets can be plotted similarly.

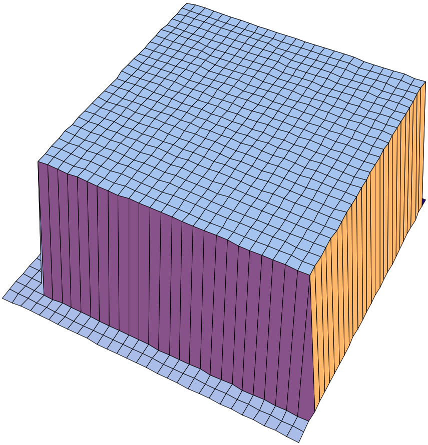


Fig. 52: Two dimensional Haar scaling function $\phi(x, y)$

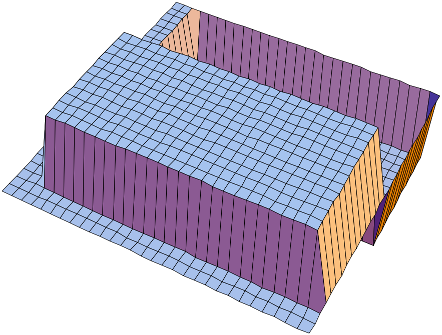


Fig. 53: Haar wavelet $\psi^{(I)}(x,y)$

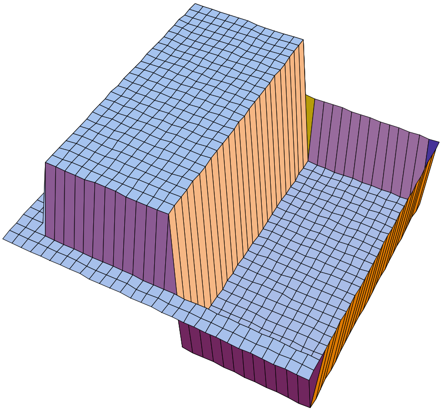


Fig. 54: Second wavelet $\psi^{(II)}(x, y) = \psi^{(I)}(y, x)$

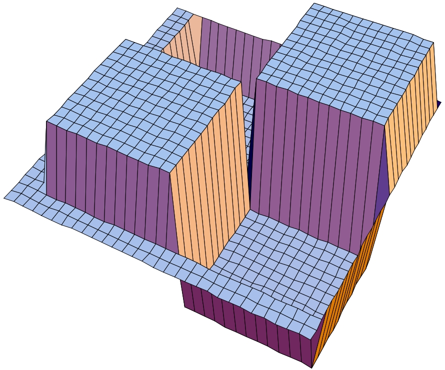


Fig. 55: Third wavelet $\psi^{(III)}(x,y)$

As example of another wavelet, here is so-called "least asymmetric wavelet" of order 8 in two dimensions :

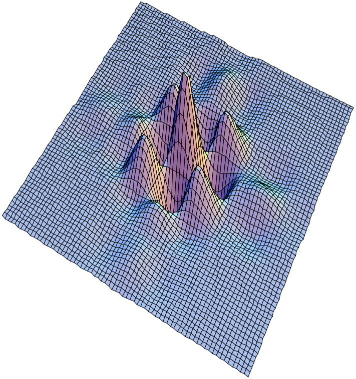


Fig. 56:  Least asymmetric wavelet of order 8