7. Random variables as observables

Definition 7. A random variable (function) **X** on a probability measure space is called an *observable*

Note all functions are assumed to be measurable henceforth.

Note that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then the family of (essentially) bounded random variables (functions on Ω) $X(\omega)$ has the structure of an integration algebra $(\mathbf{A}, E)$, with

$$E(X) = \int X(\omega) d\mathbb{P}(\omega).$$

In probability we interpret Ω as the space of all possible outcomes, and for each outcome ω, we define $X(\omega)$ to be the value of the observable X given the outcome $\omega \in \Omega$.

We can call $X(\omega)$ an observable.

We may study the integration algebra directly, since it determines the structure of the probability space up to isomorphisms (measure-preserving transformations).

Note that the algebra **A** of observables is commutative, since $X(\omega)Y(\omega) = Y(\omega)X(\omega)$.

Generalization: Consider that in quantum mechanics observables also form an algebra which is more general, since it is non-commutative. But each observable still has an expected value.

Example 3. Consider the motion of a particle on the line $\mathbb{R}^1$. Let $\mathcal{H} = L^2(\mathbb{R})$. If $\psi(x) \in \mathcal{H}$ and $\|\psi\| = 1$ then ψ represents a *probability amplitude* that particle is located at x. Also $|\psi(x)|^2$ represents the probability that particle is located at x.

The operator $B = M_x$ (multiplication by x) on $L^2(\mathbb{R})$ is the *position operator.* If particle is in state $\psi(x)$, then its expected position is

$$\langle \psi, B\psi \rangle \ = \int_{-\infty}^{\infty} dx\, \overline{\psi}(x) B\psi(x) = \int_{-\infty}^{\infty} dx\, \overline{\psi}(x) M_x \psi(x)$$

$$= \int_{-\infty}^{\infty} \overline{\psi}(x) x \psi(x) dx = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx$$

The operator $C = \frac{1}{i}\frac{\partial}{\partial x}$ (differentiation with respect to x) on $L^2(\mathbb{R})$ is the *momentum operator*. The expected momentum is

$$\langle \psi, C\psi \rangle = \int_{-\infty}^{\infty} dx\, \overline{\psi}(x) \frac{1}{i} \frac{\partial}{\partial x} \psi(x)$$

$$= -i \int_{-\infty}^{\infty} dx\, \overline{\psi}(x) \psi'(x).$$

Generally any observable quantity corresponds to a self-adjoint operator A on $\mathcal{H}$, and its expectation value (in a given state $\psi \in \mathcal{H}$) is

$$E(A) = \langle \psi, A\psi \rangle.$$

8. Random variables as observables, II

We now view a random variable $X(\omega)$, $\omega \in \Omega$ as having a value for each outcome in the universe Ω of outcomes. In other words, X has a value for each of the possible 'worlds' (possible ω representing our reality) that we are in.

We have seen that just know expectation values of an algebra of random variables on Ω is sufficient to determine the structure of Ω.

Henceforth we will focus on the algebra structure of the (bounded) random variables **A** $= \{X(\omega) : X$ is a RV on $\Omega\}$.

Now consider $\mathcal{H} = L^2_{\mathbb{C}}(\Omega)$. If $\psi(\omega) \in \mathcal{H}$, then $\rho(\omega) = |\psi^2(\omega)| \in L^1_{\mathbb{R}}(\Omega)$, and represents a probability distribution on the possible outcomes $\omega \in \Omega$. If $X \in$ **A**, then in probability

$$E(X) = \int_\Omega X(\omega)\rho(\omega)d\omega$$

$$= \int_\Omega \overline{\psi}(\omega)X(\omega)\psi(\omega)dP(\omega)$$

$$= \langle\psi(\omega), X(\omega)\psi(\omega)\rangle = \langle\psi, X\psi\rangle.$$

Thus we can interpret a state $\psi(\omega)$ to be a probability distribution $\rho(\omega) = |\psi(\omega)|^2$ on Ω.

It represents our current knowledge of the world as a probability distribution $\rho(\omega)$ on Ω.

9. An alternative interpretation of $\psi(\omega)$

Note that if we have a state ψ, it forms a linear functional ϕ on the set of observables $X \in$ **A**. Specifically, define

$$\phi(X) = E_\psi(X) = \int X(\omega)|\psi|^2(x)dP(\omega) = \langle \psi, X\psi \rangle.$$

Thus a state ψ can be interpreted as a linear functional $\phi :$ **A** $\longrightarrow \mathbb{C}$.

Definition 8. More generally, we will define a *state* to be *any* bounded linear functional ϕ on **A**.

10. Summary of commuting observables

So far we have showed that studying an algebra **A** of random variables on a probability space Ω is entirely equivalent to knowing the algebra structure of **A**, i.e. knowing XY and $aX + bY$ (for $a, b \in \mathbb{C}$) if we know X and Y.

We call the algebra **A** of observables *commuting* because $XY = YX$ if $X, Y \in$ **A**.

We have seen that if $\psi(\omega) \in L^2(\Omega)$, then $|\psi|^2(\omega) \in L^1(\Omega)$ and represents a *new* probability distribution on Ω.

Note that the expectation with respect to this new distribution is

$$E_\psi(X) = \int_\Omega X(\omega) \underbrace{|\psi|^2(\omega) dP(\omega)}_{P_\psi(\omega)}.$$

Note that this is equivalent to definining a *new* measure on Ω

$$P_\psi = |\psi|^2(\omega)dP(\omega),$$

i.e., Radon-Nikodym derivative

$$\frac{dP_\psi}{dP} = |\psi|^2(\omega).$$

We have defined the state ψ through its expectation E_ψ above as a linear functional on $X \in$ **A**.

More generally, we have now defined a *state* as *any* linear functional ϕ : **A** $\rightarrow \mathbb{C}$.

But note that this E_ψ is an *expectation* on the algebra **A** of random variables (i.e. satisfies the 4 properties defined earlier).

Thus we have a

Proposition. Given a probability space Ω, an algebra **A** of random variables on Ω, and $\psi(\omega) \in L^2(\Omega)$ with $\|\psi\| = 1$, there is a unique expectation E_ψ defined on **A** given by

$$E_\psi(X) = \int_\Omega X(\omega)|\psi|^2(\omega)dP(\omega).$$

Note that this expectation can be interpreted as giving the average value of any observable X given the underlying state ψ on Ω.

11. Non-commuting observables

Recall from above we have an algebra **A** of observables X together with an expectation E on the algebra (we denote $E = E_\psi$ for now). These form a *integration algebra* defined above.

We define the total measure of the integration algebra to be $E(1)$ (i.e. expectation of the function $X = 1$).

If $E(1) = 1$ (i.e. total measure is 1) we will call **A** a *probability algebra*.

Thus a probability algebra **A** is just a generalization of the collection of random variables on a probability space Ω.

Let's generalize the above ideas to the case where observables do not commute, i.e. we have an algebra **A** of observables where $XY \neq YX$.