Lecture 3.

1. Bounded Linear Transformations

Let \( V_1 \) and \( V_2 \) be two normed linear spaces. Let \( T : V_1 \to V_2 \).

Then \( T \) is a \textit{bounded linear transformation} (BLT) if

(i) \( T(v_1 + v_2) = T(v_1) + T(v_2) \)
(ii) \( T(\alpha v_1) = \alpha T(v_1) \) [linearity]
(iii) \( \|Tv_1\| \leq C \|v_1\| \quad \forall v_1 \in V \) and some \( C > 0 \) [boundedness]

The \textit{smallest} \( C \) which makes (iii) true is called \( \|T\| = \text{norm of } T \).

\textbf{Ex 1:} Let \( V_1 = \mathbb{R}^2 \); \( V_2 = \mathbb{R}^2 \)
\[ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow \|v\| = \sqrt{v_1^2 + v_2^2} \]

For \( v \in V_1, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \)

let
\[ T = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 - v_2 \\ v_1 + v_2 \end{pmatrix} \]

Thus if \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \)
\[ Tv = \begin{pmatrix} 3v_1 - v_2 \\ v_1 + v_2 \end{pmatrix} \]

Note \( (v_1 - v_2)^2 \geq 0 \Rightarrow v_1^2 + v_2^2 - 2v_1v_2 \geq 0 \)
\[ v_1^2 + v_2^2 \geq 2v_1v_2 \]
\[ \Rightarrow 4v_1v_2 \leq 2(v_1^2 + v_2^2) \]
So

\[ \|Tv\| = ((3v_1 - v_2)^2 + (v_1 + v_2)^2)^{1/2} \]

\[ = (10v_1^2 + 2v_2^2 - 4v_1v_2)^{1/2} \]

\[ \leq (10v_1^2 + 2v_2^2 + 2(v_1^2 + v_2^2))^{1/2} \]

\[ = (12v_1^2 + 4v_2^2)^{1/2} \]

\[ \leq (12(v_1^2 + v_2^2))^{1/2} \]

\[ \leq \sqrt{12} (v_1^2 + v_2^2)^{1/2} \]

\[ = \sqrt{12} \|v\|. \]

\[ \Rightarrow T \text{ is bounded} \]

[In fact can show some way that any linear \( T \) is bounded in finite dimensional vector spaces:]

**Theorem 1:** Let \( T : V_1 \rightarrow V_2 \) be a linear transformation. Then \( T \) is bounded iff \( T \) is continuous.

**Proof:** (\( \Rightarrow \)) If \( T \) bounded, then must show \( T \) continuous.

To show this: let \( \{v_n\}_{n=1}^{\infty} \subset V_1 \), with

\[ v_n \rightarrow v \] (i.e., \( \|v_n - v\| \rightarrow 0 \))

[distance of \( v_n \) to \( v \) goes to 0 ]

Must show

\[ T v_n \rightarrow Tv \]
normally we write $T(v) = Tv$ for transformation $T$. ]

OR: must show $\|Tv_n - Tv\| \to 0$  (**)

But:

$$\|Tv_n - Tv\| = \|T(v_n - v)\| \leq C \|v_n - v\|,$$

proving (*).

( $\iff$ ) proved similarly (using inverse of above - i.e., assume $T$ unbounded and show $T$ is discontinuous).

Countable and uncountable sets:

Def. 1 A set $A$ is countable if there is a one to one correspondence between $A$ and $\mathbb{N} = $ natural numbers. It is uncountable if there is no such correspondence.

Ex. 2 $\mathbb{N}$ is countable; $\mathbb{Z} = $ integers are countable; $\mathbb{Q} = $ rationals are countable; however, $\mathbb{R}$ is uncountable.

2. Review of Measure Theory

Motivation: we wish to define sizes of sets other than intervals on $\mathbb{R}$.

Perhaps find an extension of notion of length to all subsets of $\mathbb{R}$.

That's impossible -- e.g., Banach-Tarski paradox

But: we can get a large collection of sets for which the notion of size makes sense: the Borel Sets ]

Construction of Borel Sets:

(1) Start with all open sets in $\mathbb{R}$. (Can show they are all countable unions of open intervals).
(2) Form all possible countable unions, intersections, complements of sets in (1).

(3) Form all possible countable unions, intersections, complements, etc. of sets in (2) etc.

Take all things in this sequence, then repeat entire process again. Continue this way till we have the Borel sets...

**Simpler definition:**

**Def. 3:** A family $\mathcal{F}$ of sub-sets in a set $U$ is a $\sigma$-field if

(i) The family is closed under complements

(ii) It is closed under countable unions.

[i.e., if a countable family of sets in $\mathcal{F}$, union also in $\mathcal{F}$]

(iii) $U \in \mathcal{F}$

**Def. 4:** The *Borel sets* are the smallest $\sigma$-field of sets containing all open intervals.

**Def. 5:** Let $\mathcal{J} =$ collection of all countably infinite unions of disjoint open intervals in $\mathbb{R}$.

**Ex:**

\[ \text{in } \mathcal{J} \]

\[ 1 \quad 2 \]

**Ex:**
etc.

**Def. 6:** for $I \in \mathcal{J}$ let

$$I = \bigcup_i (a_i, b_i)$$

(where the intervals $(a_i, b_i)$ are assumed disjoint).

Then define

$$\text{measure of } I = \mu(I) = \sum_{i=1}^{\infty} (b_i - a_i)$$

[i.e., we have extended notion of length somewhat]

**Def. 7:** For $B$ a Borel Set, define

$$\mu(B) = \text{measure of smallest } I \in \mathcal{J} \text{ containing } B$$

$$= \inf_{I \in \mathcal{J}} \mu(I)$$

The measure $\mu(B)$ is the *Lebesgue measure* of $B$.

**Theorem 2:**

(i) $\mu(\emptyset) = 0$

(ii) If $A_n$ is Borel for all $n$, $n = 1, 2, 3...$ and all $A_n$ are disjoint, then

$$\mu(\bigcup_n A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

[Note pf. of (i) is trivial, (ii) will not be proved]
**Def. 8.** Any nonnegative function on sets is a *measure* if it satisfies properties (i) and (ii) above.

3. **Borel Functions:**

**Def. 9:** $f : \mathbb{R} \to \mathbb{R}$ is a *Borel function* if $f^{-1}((a, b)) \in \mathcal{B}$ (Borel sets) $\forall$ intervals $(a, b)$.

**Note:** We allow values of $\pm \infty$ for our $f$'s

[This will be largest class of functions for which we will be able to define an integral.]

**Proposition 3:** $f$ is Borel iff $f^{-1}(B) \in \mathcal{B} \quad \forall B \in \mathcal{B}$.

**Proof:** Left as exercise.

**Lim sup, lim inf**

**Recall:** If $A \subset \mathbb{R}$, a *limit point* $x$ of $A$ is a point which has elements of $A$ arbitrarily close to it, not counting $x$ (also known as cluster point).

**Def. 10:** If $A \subset \mathbb{R}$ is an $\infty$ set which is bounded (i.e., set is located in some finite interval), then lim sup $A = largest$ limit point of $A$

$\liminf A = smallest$ limit point of $A$

**Ex. 2:** $A = \{a_n\}_{n=1}^{\infty}$

$$a_n = (-1)^n + \frac{1}{n}$$

**|**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>***</td>
<td>*</td>
<td>**</td>
<td>*</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\frac{3}{2}$</td>
</tr>
</tbody>
</table>

23
limit points of $A = \lim$ points of $\{a_n\} = \pm 1$

$\limsup A = 1 \quad \liminf A = -1$

Notation: $\limsup a_n = \limsup A$;
$\liminf a_n = \liminf A$

Sets can get more complicated -- this is a simple example

If $A$ is unbounded above, $\limsup A = \infty$
If $A$ is unbounded below, $\limsup A = -\infty$.

[note that as above, all this applies to sequences -- each sequence $\{a_n\}$ can be considered as set $A$ consisting of its own elements]

**Measure theory (continued):**

Theorem 4: (a) If $f, g$ are Borel measurable, so is $f + g, fg, \max (f, g)$, and $\min (f, g)$. Also, $f$ Borel $\Rightarrow \lambda f$ Borel $\forall \lambda \in \mathbb{R}$.

[can prove part (a) by approximation below by step functions and using part (b) below]

(b) if $f_n$ are Borel measurable, and $f_n \rightarrow f$, then $f$ is Borel measurable.