Lecture 5.

1. Abstract Measure Theory

Recall: A collection $\mathcal{A}$ of subsets of a set $M$ is a $\sigma$-field (or $\sigma$-algebra) if

(i) $A_i \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow \sim A \in \mathcal{A}$

(iii) $M \in \mathcal{A}$

Def 1: A measure $\mu$ on the $\sigma$-field $\mathcal{A}$ assigns a "size" $\mu(A)$ to each $A \in \mathcal{A}$, such that

(i) $\mu(\emptyset) = 0$

(ii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $A_i$ disjoint

(countable additivity)

$(M, \mu) = \text{measure space}$

Def 2: $\mu$ is a $\sigma$-finite measure on set $M$ if

$M = \text{countable union of sets } A_i \text{ with } \mu(A_i) < \infty \forall i.$

[We'll always assume $\mu$ $\sigma$-finite; otherwise things get too big].

Def 3: If $M = \text{measure space and } f : M \rightarrow \mathbb{R}, f \text{ is measurable }$ if

$f^{-1}((a, b)) = \text{measurable set } \forall \text{ intervals } (a, b)$.

As before, define

$$\int f(x) d\mu = \lim_{n \rightarrow \infty} \sum_{m} \mu\left(f^{-1}\left(\frac{m}{n}, \frac{m+1}{n}\right)\right) \frac{m}{n}$$

i.e., integral of function identical to Lebesgue integral on $\mathbb{R}$, using $\mu$ a new measure on $M$.

$\Rightarrow$ get
Monotone Convergence Theorem\hspace{1cm} Stated and proved

Dominated Convergence Theorem\hspace{1cm} exactly as in use of

Riesz-Fisher theorem\hspace{1cm} Lebesgue measure.

\[ L^1(M) = \{ f : \int |f(x)|d\mu < \infty \} \]

[unless specified, integral always means over whole space]

2. Multiple integrals:

Let \((M, \mu)\) and \((N, \nu)\) be measure spaces. Define the product set

\[ M \times N = \{(x, y) : x \in M, y \in N\}. \]

Let \(\mathcal{A}(M) = \sigma\)-field of sets on \(M\) on which \(\mu\) defined

\(\mathcal{A}(N) = \sigma\)-field of sets on \(N\) on which \(\nu\) defined

Define \(\mathcal{A}(M) \times \mathcal{A}(N) = \text{smallest } \sigma\)-field on \(M \times N\) which contains all product sets of the form

\[ A \times B = \{(a, b) : a \in A, b \in B\} \]

with \(A \in \mathcal{A}(M), B \in \mathcal{A}(N)\).

**Theorem:** There exists a unique measure \(\mu \times \nu\) on \(\mathcal{A}(M) \times \mathcal{A}(N)\) with the property

\[(\mu \times \nu)(A \times B) = \mu(A)\mu(B)\]

for \(A \in \mathcal{A}(M), B \in \mathcal{A}(N)\).

**Def:** The measure \(\mu \times \nu\) is the product measure of \(\mu\) and \(\nu\).

**Fubini's theorem:** If \(f(x, y)\) = measurable function on \(M \times N\), then

\[ \int_M d\mu(x) \left( \int_N d\nu(y) f(x, y) \right) = \int_{M \times N} d(\mu \times \nu) f(x, y) \]
if any of these three integrals converges absolutely.

**Ex 1:** $M = N = \mathbb{R}$; $\mu = \nu = \text{Lebesgue measure}$. Then
\[
\int d\mu(x) \left( \int d\nu(y) f(x, y) \right) = \int d\nu(y) \int d\mu(x) f(x, y) = \int d\mu(x) \int d\nu(y) f(x, y)
\]
allowing interchange of order of integration.

3. **Singular measures:**

**Def 4:** Two measures $\mu, \nu$ on space $M$ are *mutually singular* if $\exists$ a set $A$ such that $\mu(A) = 0$ and $\nu(\sim A) = 0$

[i.e., $\nu$ lives on $A$, $\mu$ lives on $\sim A$; supported in different locations]

**Def 5:** $\nu$ is *absolutely continuous* w.r.t. $\mu$ if $\nu(A) = 0$ whenever $\mu(A) = 0 \quad \forall A \in \mathcal{A}$.

[compare with Lebesgue definitions — these are extensions]

[Following theorem connects above definition of absolute continuity with absolute continuity w.r.t. Lebesgue measure, mentioned earlier.]

4. **Radon-Nikodym Theorem:**

$\nu$ is absolutely continuous w.r.t. $\mu$ iff
\[
\nu(A) = \int_A f(x) \, d\mu
\]

for some measurable $f(x)$ and $\forall A \in \mathcal{A}$.

Lebesgue Decomposition Theorem carries over here (with appropriate generalizations of notion of singular and absolutely continuous measures).
5. \( \frac{2}{3} \) Arguments:

Ex:

**Theorem:** Let \( V = C[a, b] \). For \( f \in V \), let

\[
\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|.
\]

[can easily check that \( \| \cdot \|_\infty \) is norm]

Then \( V \) is complete as a metric space.

**Proof:** Must show that if \( f_n \in V \) are a Cauchy sequence, then \( f_n \)
\( \rightarrow_{\infty} f \), where \( f \in V \).

Let \( \{ f_n \}_{n=1}^{\infty} \) be Cauchy. Then \( \forall \epsilon > 0 \), \( \|f_n - f_m\|_\infty < \epsilon \) for \( n, m \)
sufficiently large, i.e., \( \sup_{x \in [a, b]} |f_n(x) - f_m(x)| < \epsilon \) for sufficiently large \( n, m \).

Fix \( x \in [a, b] \). Then

\[
|f_n(x) - f_m(x)| < \epsilon
\]

for sufficiently large \( n, m \) i.e., \( \{ f_n(x) \}_{n=1}^{\infty} = \) Cauchy sequence of numbers [now for fixed \( x \)].

Thus, since \( \mathbb{R} \) complete, for each \( x \)

\[
f_n(x) \xrightarrow{n \to \infty} \text{something, call it } f(x).
\]

Will show: \( \|f_n(x) - f(x)\|_\infty \xrightarrow{n \to \infty} 0 \).

Let \( \epsilon > 0 \); let \( N \) be such that for \( n, m \geq N \)

\[
\Rightarrow \|f_n - f_m\| < \epsilon.
\]

Thus, if \( m > N \) is fixed,

\[
\|f - f_m\|_\infty = \sup_{x \in [a, b]} |f(x) - f_m(x)|
\]
\[
= \sup_{x \in [a,b]} \lim_{n \to \infty} |f_n(x) - f_m(x)|
\]

\[
\leq \sup_{x \in [a,b]} \sup_{n > N} |f_n(x) - f_m(x)|
\]

[since a limit of sequence of numbers always \( \leq \) their sup]

\[
= \sup_{n > N} \sup_{x \in [a,b]} [f_n(x) - f_m(x)]
\]

[since suprema are commutative]

\[
= \sup_{n > N} \| f_n - f_m \|_{\infty} < \epsilon.
\]

\[\Rightarrow \| f - f_m \| \to 0\]

\[\Rightarrow f_m \to f\]

[Thus, the Cauchy sequence does converge to \( f \). But is \( f \in C[a, b] \)?]

**Need:** [done after this Lemma]

**Lemma.** If \( f_n(x) \) are continuous on \([a, b]\) and \( \| f_n - f \|_{\infty} \to 0 \), then \( f \) is continuous.

**Pf:** To show \( f \) is continuous, we'll use \( \epsilon-\delta \) definition.

Let \( x \in [a, b] \); pick \( \epsilon > 0 \). Want \( \delta > 0 \) such that if
\[
|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.
\]

To find \( \delta \), pick \( n \) such that
\[
\| f_n - f \|_{\infty} < \epsilon/3.
\]

Since \( f_n \) continuous, pick \( \delta \) such that
\[
|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\epsilon}{3}.
\]

Thus, if \( |x - y| < \delta \),

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\[ |f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \]
\[
\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon .
\]

\[ \Rightarrow \quad f(x) \text{ is continuous.} \quad \square \]

6. Hilbert Space

[Recall \( \mathbb{C} = \) complex numbers]

**Def:** Let \( V \) be a complex vector space (i.e., if \( v \in V \) and \( c \in \mathbb{C} \), \( cv \) defined).

Assume that \( \forall x, y \in V \), there is a number
\[ \langle x, y \rangle \in \mathbb{C}. \]

[called an inner product]

such that

(a) \( \langle x, x \rangle \geq 0 \) \quad (\langle x, x \rangle = 0 \iff x = 0)  
(b) \( \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \)  
(c) \( \langle x, \alpha y \rangle = \alpha \langle x, y \rangle \)  
(d) \( \langle x, y \rangle = \overline{\langle y, x \rangle} \)

where \( \overline{\cdot} \) denotes complex conjugate, i.e., \( \overline{(a + bi)} = a - bi \).

Then \( V \) is an inner product space (IPS) [note \( \langle \cdot, \cdot \rangle \) is like a dot product]

**Ex:** Let \( V = \{(a_1, a_2, a_3) : a_i \in \mathbb{C}\} \)  

[ 3 - d vector space]

\[ \Rightarrow \quad \text{let } v_1 = (a_1, a_2, a_3) \]
\[ v_2 = (b_1, b_2, b_3) \]
\[ \Rightarrow \langle v_1, v_2 \rangle = \sum_{i=1}^{3} a_i b_i. \]

[can check it's an inner product space]

**Ex:** Let \( V = C[0, 1] \).

Then if \( f_1, f_2 \in V \),
\[ \langle f_1, f_2 \rangle = \int_0^1 f_1(x)f_2(x)dx. \]

[again can check properties of inner product]

**Def:** If \( V \) is an IPS, \( x, y \in V \) are **orthogonal** if \( \langle x, y \rangle = 0 \).

[perpendicularity]

**Def:** If \( \{x_i\}_{i=1}^k \subset V \) (\( k \) can be \( \infty \)) is a set of vectors which are orthogonal to each other and such that \( \langle x_i, x_i \rangle = 1 \ \forall i \), the \( x_i \) are called an **orthonormal set** of vectors.

**Def:** For a vector \( v \in V \), define
\[ \|v\| = \sqrt{\langle v, v \rangle}. \]

**Theorem:** Let \( \{x_i\}_{i=1}^k \) be an orthonormal set (\( k < \infty \)). Then \( \forall x \in V \),
\[ \|x\|^2 = \sum_{i=1}^{k} |\langle x, x_i \rangle|^2 + \left\| x - \sum_{i=1}^{k} \langle x, x_i \rangle x_i \right\|^2 \]

[Proof in R&S. See it!]

**Corollary** (Bessel inequality): Let \( \{x_i\}_{i=1}^k \) (\( k < \infty \)) be an orthonormal set in \( V \). Then \( \forall x \in V \),
\[ \|x\|^2 \geq \sum_{i=1}^{k} |\langle x, x_i \rangle|^2 \]

**Pf:** Clear from theorem.

**Corollary** (Schwarz inequality):

\[ |\langle x, y \rangle| \leq \|x\| \|y\| \quad \forall x, y \in V \]

**Pf:** Simple - in R & S.

**Theorem:** If \( V \) is an IPS, and

\[ \|x\| = \sqrt{\langle x, x \rangle} , \]

then \( V \) with norm \( \|x\| \) satisfies the properties of being a normed linear space.

**Pf:** Straightforward computations.

Thus, we also have metric

\[ \rho(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle} \]

[We know this is a metric on any NLS, so it must be here too]