Suggestions for PS 10:

1. IV.10: (a) try a countable set of elements of X.
(b) let $\mathcal{C}$ be an open cover of X. How much of X does a set in $\mathcal{C}$ cover? Does this extend into a finite subcollection which covers X?

2. IV.11: (a) Does the sequence argument work if we translate to nets?
(b) For two limits x and y, find open sets which separate them.

3. IV.15: You can use previous results about such series. To show $\mathcal{A}$ is an algebra note every f, g $\in \mathcal{C} = C([0,2\pi])$ have series

$$f = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \qquad g = \sum_{n=-\infty}^{\infty} b_n e^{inx}.$$

If f, g $\in \mathcal{A}$ the sums range over non-negative n. For f, g $\in \mathcal{A}$ let f_N and g_N be their partial sums. Show for each k

$$\left| \int (fg - f_N g_N)\, e^{ikx}\, dx \right| \leq \| f_N \|_2 \, \| g_N - g \|_2 + \| g \|_2 \, \| f_N - f \|_2$$

(try Hölder), so fg $\in \mathcal{A}$. To show $\mathcal{A}$ closed, show if f_n is Cauchy in $\mathcal{A}$ (in the uniform convergence topology of continuous functions), then its limit is in $\mathcal{A}$. How to show $\mathcal{A}$ separates points (note all e^{ikx} for k > 0 are in $\mathcal{A}$)?

4. IV.16: We want the version of S-W theorem in Theorem IV.9. Why can we assume B is not of the form $\{f \in C_{\mathbb{R}} \mid f(x_0) = 0\}$ for any x_0?. To show B = $C_{\mathbb{R}}(X)$, show B is closed so it suffices to show it is dense in $C_{\mathbb{R}}(X)$. Does the text proof work if we do not assume 1 $\in$ B? Does Lemma 2 of the Appendix to IV.3 hold? We would need to prove first (see proof in the book) that the algebra $\mathcal{A}'$ of polynomials P(x) on [-1,1] without constant terms can still approximate |x|. Note |x| can be approximated by $\mathcal{A}$ if the latter is all polynomials. Try to replace $P_n(x)$ by $P_n(x) - P_n(0)$. Thus the proof of Lemma 2 goes through without 1 in B.

For Theorem IV.12, note 1 $\in$ B is only assumed for constructing f_{xy}. Show you need to establish that without 1 it is still possible for any h $\in C_{\mathbb{R}}(X)$ to find a $f_{xy} \in$ B with $f_{xy}(x)$ = h(x), $f_{xy}(y)$ = h(y). Note given x, y, if a is a function in B which has different non-zero values there (why does such a function exist?), we can multiply a by a constant so a(x) = 2. If b(x) = $a^2(x)$, why is $a(x)b(y) - a(y)b(x) \neq 0$? Show then there are constants c_i with

(**) $\qquad c_1 a(x) + c_2 b(x) = h(x), \qquad c_1 a(y) + c_2 b(y) = h(y)$?

5. IV.19: Consider $\int_{-1/2}^{1/2} s_k(x - y)\,(f(y) - f(x))\,dy$, and show this goes to 0 uniformly by writing

$$\int_{-1/2}^{1/2} s_k(x - y)\,(f(y) - f(x))\; dy = \left(\int_{A_\delta} + \int_{\sim A_\delta} \right) s_k(x - y)\,(f(y) - f(x))\,dy\,,$$

where $A_\delta = (x-\delta, x+\delta)$. Now show for fixed δ the second integral can be made small for large k (uniformly in x of course), while the first integral is bounded for all k by noticing that since f is continuous on a compact interval, it is uniformly continuous (this implication may be assumed). Why does this show, uniformly in x, $\sup_{y\in A_\delta}|f(y) - f(x)| \underset{\delta\to 0}{\longrightarrow} 0$? Shows thus for any $\epsilon > 0$ there is a δ and a k_0 such that for $k > k_0$, both integrals are absolutely less than $\epsilon/2$. Finally why does $\int_{-1/2}^{1/2} s_k(x-y)\, f(y)\, dy \underset{k\to\infty}{\longrightarrow} f(x)$ uniformly?

6. IV.20: Set $s_k(x) = (1-x^2)^k/I_k$, and use the previous result; show $\int_{-1}^{1} s_k(x-y)\, f(y)\, dy$ is a polynomial.

7. IV.31: Note in the proof of Theorem IV.16 $C(X)$ should be $C_{\mathbb{R}}(X)$. Assume ℓ'_+ and ℓ'_- is another pair of positive functionals with $\ell = \ell'_+ - \ell'_-$, with $\|\ell\| = \ell'_+(1) + \ell'_-(1)$. Show for any non-negative f(x),

$$\ell(f) = \ell'_+(f) - \ell'_-(f) \le \ell'_+(f)$$

so

$$\sup_{0\le h\le f} \ell'_+(h) \ge \sup_{0\le h\le f} \ell(h) = \ell_+(f).$$

Why does it follow for $f \ge 0$, $\ell'_+(f) \ge \ell_+(f)$? Thus $\ell'_+ - \ell_+$ is positive. However

$$\ell'_+(1) + \ell'_-(1) = \|\ell\| = \ell_+(1) + \ell_-(1).$$

Why does this show $\ell'_\pm(1) = \ell_\pm(1)$? Show thus for any $0 \le f \le 1$

$$(\ell'_+ - \ell_+)f \le (\ell'_+ - \ell_+)(1) = 0 .$$