Suggestions for problem set 11:

1. IV.41: $\mathcal{M}(X)$ are the complex measures on X. By Theorem IV.17 $\mathcal{M}(X) = C(X)^*$, with the same norms. The convergent linear combinations of point measures are measures $\sum_{i=1}^{\infty} c_i \delta_{x_i}$ with c_i complex and $\sum_{i=1}^{\infty} |c_i| < \infty$. To show this set A is closed in $\mathcal{M}(X)$ is equivalent to showing A is complete as an NLS (why?), and this is equivalent to showing if $\mu_j \in A$ and $\sum_{j=1}^{\infty} \| \mu_j \| < \infty$, then $\sum_{j=1}^{M} \mu_j$ converges as $M \to \infty$. Write

$$\mu_j = \sum_{i=1}^{\infty} c_{ji}\, \delta_{x_{ji}}$$

Show

$$\| \mu_j \| = \sup_{\| f \| = 1} \int f(x)\, d\mu_j = \sup_{\| f \| = 1} \sum_{i=1}^{\infty} c_{ji}\, f(x_{ji}) = \sum_{i=1}^{\infty} |c_{ji}|$$

with suprema over all continuous $f(x)$ whose maximum absolute value is 1. If $\sum_{j=1}^{\infty} \| \mu_j \| < \infty$, why does it follow that $\sum_{j,i} |c_{ji}| < \infty$? Thus show

$$\sum_{j=1}^{M} \mu_j = \sum_{j=1}^{M} \sum_{i=1}^{\infty} c_{ji}\, \delta_{x_{ji}}$$

converges in norm to $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ji}\, \delta_{x_{ji}} \in A$.

3. II.17: What can we conclude if a space has an infinite collection of disjoint sets of the same positive measure?

4. II.18: We want to prove the statement from class. Show (at least for τ a positive integer):

$$\frac{1}{\tau} \int_0^{\tau} f(T_t x)\, dt = \frac{1}{\tau} \int_0^1 \sum_{N=0}^{\tau-1} U_1^N f(T_t x)\, dt.$$

Now show if P $=$ projection onto J $= \{\psi |\ U_1 \psi = \psi\}$, taking L^2 norms with respect to x,

$$\left\| \frac{1}{\tau} \int_0^1 \sum_{N=0}^{\tau-1} U_1^N f(T_t x)\, dt - \int_0^1 Pf(T_t x)\, dt \right\|_2$$

$$\leq \int_0^1 \left\| \frac{1}{\tau} \sum_{N=0}^{\tau-1} U_1^N f(T_t x) - Pf(T_t x) \right\|_2 dt. \tag{1}$$

How does unitarity of U_1 help show for each t, the integrand goes to 0? On the other hand show for each t, the last integrand is bounded by a fixed constant (try to bound each term in the norm of (1) individually; why is $\| f(T_t x) \|_2 = \| f(x) \|_2$? Note norms are wrt x). Use dominated convergence to take the limit. You can also show the Corollary to the MET since if T_t is ergodic for each t, J $=$ constant functions (why?). Thus show for any f, Pf $= \langle 1, f \rangle$, where 1 is the function identically 1.

5. II.20: If θ is irrational, move to the unit interval, where Tx = x + θ mod 1 (if $x > 1$ take its fractional part). Let $\phi_n = e^{2\pi inx}$. What is $U\phi_n$? If $f = \sum_n a_n\phi_n$ and Uf = f, why is $e^{2\pi in\theta} = 1$ for any n with $a_n \neq 0$? Conversely if θ is rational, why are there non-constant f of the above form preserved by U?

6. II.21: Why does uniform equicontinuity follow from the fact T is measure preserving and f is uniformly continuous? You can use the fact that any continuous function on a compact space is uniformly continuous, i.e., given $\epsilon > 0$, there is $\delta > 0$ such that for any x,y with $\rho(x,y) < \delta$, $|f(x) - f(y)| < \epsilon$. Show given $\epsilon > 0$, if $\rho(x, y) < \delta$

$$|M_N f(x) - M_N f(y)| \leq \frac{1}{N}\sum_{n=0}^{N-1}|f(T^n x) - f(T^n y)| < \epsilon.$$

so M_N are uniformly equicontinuous. How does Theorem 1.27 apply?

9. VI.3a: If Tx is the limit of this sequence for each x, is T bounded? How does Theorem VI.1 apply?