Suggestions for PS 12:

4. VI.9:

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that $||A|| = \sup_{||x||=1} |\langle Ax, x\rangle|$

(a) Assume all suprema over ϕ, ψ, x are done over the unit ball, e.g., with $|\phi| \leq 1,\ |\psi| \leq 1,$ $|x| \leq 1,$ etc., unless stated otherwise. First you can show by multiplying ψ by a phase constant $e^{i\theta}$ that $\sup_{\psi,\phi}|\langle\psi, A\phi\rangle| = \sup_{\psi,\phi} \mathrm{Re}\langle\psi,A\phi\rangle$. First assume A is positive. Then show

$$\mathrm{Re}\langle\psi, A\phi\rangle = \frac{1}{4}[\langle\psi+\phi, A(\psi+\phi)\rangle - \langle\psi-\phi, A(\psi-\phi)\rangle],$$

$$\leq \frac{1}{4}\left(\|\psi+\phi\|^2 \sup_x\langle x, Ax\rangle + \|\psi-\phi\|^2 \sup_x\langle x, Ax\rangle\right)$$

$$= \frac{1}{4}\left(2\|\phi\|^2 + 2\|\psi\|^2\right) \sup_x\langle x, Ax\rangle.$$

which will prove the result.

(b) You can find a 2×2 matrix for a counterexample