Suggestions for PS 13

1. VI.18: If U is a partial isometry from Ker $U^\perp$ to Ran U, why does it hold that U^* is a PI from Ran U to Ker $U^\perp$ and is the inverse of the map U on Ran U? Note that both Ker U and Ran U are closed, and similarly for U^*. Why is Ker U = Ran $U^{*\perp}$ (use a previous lemma; the same then holds replacing U by U^*). For the inverse property for U^*, try to show that if $z \in$ Ker $U^\perp$ and $x = Uz$ then $U^*x = z$.

If $P_i = U^*U$ and $P_f = UU^*$ are projections, why is U a PI from Ran P_i to Ran P_f? If $x \in$ Ran P_i, show $\|Ux\| = \|x\|$. Show Ker U = Ker P_i, because U^* never vanishes on Ran U (why)?

For uniqueness in Thm. VI.10, if $|A|x = 0$ how does it follow Ax = 0? Note if $|A|x$ = 0 then A*Ax = 0, so Ker $|A| \subset$ Ker A. If there were another partial isometry U' satisfying the properties of U, why does it suffice to show $Ux = U'x$ for $x \in$ Ker $A^\perp$? By $U|A| = U'|A|$, it suffices to show $\overline{\text{Ran}\,|A|}$ contains Ker $A^\perp$. How can you use $\overline{\text{Ran}|A|}$ = Ker $|A|^\perp$?

3. Proof of Theorem VII.1:
Only some parts of the R&S proof are clear. In particular you can assume $\phi : \mathcal{P}(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ (with $\mathcal{P}(\sigma(A))$ polynomials restricted to $\sigma(A)$) is an isometry. Why does it extend to a bounded map on the continuous functions $C(\sigma(A))$? How do properties (a), (b), (c) follow (in particular, why do they hold for polynomials?) Why do these properties uniquely determine ϕ on $\mathcal{P}(\sigma(A))$? Prove (d), (f), and (g). You can use the fact that (e) will follow from Problem VII.8 (to be done).

4. Proof of Theorem VII.2:
Given $f \in \mathcal{B}(\sigma(A))$ and bounded, show that for any two bounded sequences $\{f_n(x)\}$ and $\{\overline{f}_n(x)\}$ of continuous functions coverging to f pointwise, the sequence $\{f_n(A) - \overline{f}_n(A)\}$ converges to 0 in the strong topology; this will show that there must be a unique strong limit to any sequence $\{f_n\}$ which converges to $f(x)$. To do this, assume that this is not so for some pair $\{f_n, \overline{f}_n\}$ and let $g_n = f_n - \overline{f}_n$. Show there would be a vector z such that $\limsup_{n\to\infty}$ $\|g_n(A)z\| > \epsilon$ for some $\epsilon > 0$. However show this would mean that $\langle g_n(A)^2 z, z\rangle = \int |g(x)|^2 d\mu_z$ fails to go to 0. Show this contradicts the dominated convergence theorem. Therefore indeed, if $f_n \to f$ pointwise, then $f_n(A)$ has a unique strong limit, which we define to be $f(A)$.

To prove property (a), you need to show that for $f, g \in \mathcal{B}(\sigma(A))$, $f(A)g(A) = (fg)(A)$. Let f_n and g_n be sequences of unifromly bounded continuous functions converging to f, g respectively. Then $f_n(A)g_n(A) = (f_n g_n)(A)$. The right hand side converges to $(fg)(A)$ strongly, and you need to show the left side converges to $f(A)g(A)$.

Show the latter follows from the general fact that if operators C_n and D_n converge strongly to C and D respectively, then C_nD_n converges strongly to CD. You can show this by looking at

$$C_nD_n - CD = C_n(D_n - D) + (C_n - C)D.$$

Show for any x $(C_n - C)Dx \underset{n\to\infty}{\longrightarrow} 0$, while

$$C_n[(D_n - D)x] \underset{n\to\infty}{\longrightarrow} 0$$

since C_n is bounded (use the uniform boundedness principle to show this) while $(D_n - D)x \underset{n\to\infty}{\longrightarrow} 0$.

Show that this combined with the previous agrument proves (a)

To prove (b) note that if $C_n \underset{n\to\infty}{\longrightarrow} C$ strongly, then $||C|| \le \limsup_n ||C_n||$. Show this by letting $||x|| = 1$, and showing $||Cx|| = \lim_{n\to\infty} ||C_nx|| \le \limsup_n ||C_n||\,||x|| \le \limsup_n ||C_n||$. This will give you (b) since for any $f \in \mathcal{B}(\sigma(A))$ there is a sequence of continuous functions f_n which converge to it and such that $||f_n||_\infty \le ||f||_\infty + \frac{1}{n}$. Thus show $||f(A)|| \le \limsup_n ||f_n(A)|| \le \limsup ||f||_\infty = ||f\|_\infty$, as desired.

Show (c) and (d) follow immediately, the latter from the strong convergence above of $f_n(A)$.

Show (e) follows directly from the above continuous approximation, and (f) does as well from the definition of a positive operator, while (f) also follows from the approximation.

5. VII.8: (a) How can you use the definition of an inverse and Theorem VII.1?
(b) Consider a continuous nonnegative function $\eta(x)$ bounded by 1 with support in $[\lambda - \epsilon, \lambda + \epsilon]$, and equal to 1 on $[\lambda - \epsilon/2,\ \lambda + \epsilon/2]$. Why does $\eta(f(A))$ have norm 1 (choose a ϕ of norm 1 such that $\|\eta(f(A))\phi\| = 1$ (or at least arbitrarily close to 1)). Note

$$\|(f(x) - \lambda)\eta(f(x))\|_\infty \le \epsilon,$$

(why?). Hence $\|(f(A) - \lambda)\eta(f(A))\| \le \epsilon$ (why?) Try letting $\psi = \eta(f(A))\phi$.
(c) If $\lambda \in$ Ran $f = f(\sigma(A))$, then $f(A) - \lambda$ can make vectors of norm 1 arbitrarily small. Does it have a bounded inverse? Conversely, if $\lambda \notin$ Ran f, use (a) to show that $\lambda \in \rho(A)$, the resolvent set.

Additional Suggestion: to prove part (b), you may also want to use the first proposition of Section VII.2 on the relationship of the support of the spectral measures and the spectrum. Note the typo in the problem. In part (b) $f(A)$ should just be f.